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A Concise Workbook for College Algebra

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2018

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Preface

This workbook is mainly based on the author's worksheets for the college algebra course (MAD-119) at QCC of CUNY. It is intended as a concise introduction to college algebra at the intermediate level.

Our motivation is to emphasize the depth of reasoning and understanding rather than multitudes of approaches to similar types of questions. We try to expose only key concepts and ideas, which will save time for practicing and reinforcing critical thinking skills which include observing patterns, identifying and analyzing problems, making logic connections, determining problem-solving strategies, and solving problems systematically. For example, only the method of undetermined coefficient was introduced for factoring trinomials. To factor the trinomial $Ax^2 + Bx + C$, where A , B and C are integers, we may use trial-and-error method to find integers m , n , p and q such that $mn = A$, $pq = C$ and $mq + np = B$. In practice, we first factor A and C , and then use the following diagram to check if $mq + np = B$ holds.

$$\begin{array}{ccc} A = mn & & C = pq \\ m & \begin{array}{c} \diagdown \\ \diagup \end{array} & p \\ n & \begin{array}{c} \diagup \\ \diagdown \end{array} & q \\ \hline np & + & mq & \stackrel{?}{=} & B \end{array}$$

This method is based on the observation that $Ax^2 + Bx + C$ can be factored into $(mx + p)(nx + q)$. Indeed, observing and making logic connections are very effective in problem-solving.

Topics are contained in 25 lessons. Each lesson corresponds to roughly one class meeting. A lesson starts with a page on concepts, formulas and examples, and ends with a list of practice problems that students are expected to be able to solve and complete in class.

This work was partially funded by the OER Conversion Grant from CUNY.

We would like to thank our colleagues and students for their feedback and support during the development of this project. In particular, we would like to thank Joseph Bertorelli, Beata Ewa Carvajal, Kwai Chiu, Lixu Li, Wenjian Liu, Nam Jong Moh, Tian Ren, Kostas Stroumbakis, Evelyn Tam, and Haishen Yao for their encouragement, support, carefully reading the draft and correcting mistakes.

Fei Ye

Summer 2018

Contents

Preface	ii
1 Linear Inequalities	1
2 Absolute Value Equations	3
3 Properties of Integral Exponents	5
4 Introduction to Functions	7
5 Linear functions	9
6 Perpendicular and Parallel Lines	11
7 Systems of Linear Equations	13
8 Factoring Review	15
9 Solve Quadratic Equations by Factoring	17
10 Multiply or Divide Rational Expressions	20
11 Add or Subtract Rational Expressions	22
12 Complex Rational Expressions	24
13 Rational Equations	26
14 Radical Expressions: Concepts and Properties	28
15 Algebra of Radicals	31
16 Solve Radical Equations	34
17 Complex Numbers	36
18 Complete the Square	38
19 Quadratic Formula	40
20 Graphs of Quadratic Functions	42
21 Rational and Radical Functions	44
22 Exponential Functions	45
23 Logarithmic Functions	47
24 Properties of Logarithms	49
25 Exponential and Logarithmic Equations	51

Lesson 1. Linear Inequalities

Properties of Inequalities

Property	Example
<p>The additive property If $a < b$, then $a + c < b + c$. If $a < b$, then $a - c < b - c$.</p>	If $x + 3 < 5$, then $x + 3 - 3 < 5 - 3$. Simplifying both sides, we get $x < 2$.
<p>The positive multiplication property If $a < b$ and c is positive, then $ac < bc$. If $a < b$ and c is positive, then $\frac{a}{c} < \frac{b}{c}$.</p>	If $2x < 4$, then $\frac{2x}{2} < \frac{4}{2}$. Simplifying both sides, we get $x < 2$.
<p>The negative multiplication property If $a < b$ and c is negative, then $ac > bc$. If $a < b$ and c is negative, then $\frac{a}{c} > \frac{b}{c}$.</p>	If $1 < 2$, then $-2 = 1 \cdot (-2) > 2 \cdot (-2) = -4$. If $-2x < 4$, then $\frac{-2x}{-2} > \frac{4}{-2}$. Simplifying both sides, we get $x > 2$.

Note: These properties also apply to $a \leq b$, $a > b$ and $a \geq b$.

Compound Inequalities

A compound inequality is formed by two inequalities with the word “and” or the word “or”. For example,

$$\begin{aligned}
 &x - 1 > 2 \quad \text{and} \quad 2x + 1 < 3. \\
 &3x - 5 < 4 \quad \text{or} \quad 3x - 2 > 10. \\
 &-3 \leq \frac{2x-4}{3} < 2, \quad \text{which means that } -3 \leq \frac{2x-4}{3} \quad \text{and} \quad \frac{2x-4}{3} < 2.
 \end{aligned}$$

Example 1.1. Solve the linear inequality
 $2x + 4 > 0$.

Solution:

$$\begin{array}{rcl}
 & 2x + 4 & > 0 \\
 \text{add } -4 & & 2x > -4 \\
 \text{divide by } 2 & & x > -2
 \end{array}$$

The solution set is $(-\infty, -2)$.

Example 1.3. Solve the compound linear inequality $x + 2 < 3$ and $-2x - 3 < 1$.

Solution:

$$\begin{array}{rcl}
 x + 2 < 3 & \text{and} & -2x - 3 < 1 \\
 x < 1 & & -2x < 4 \\
 x < 1 & \text{and} & x > -2
 \end{array}$$

That is $-2 < x < 1$. The solution set is $(-2, 1)$.

Example 1.5. Solve the compound linear inequality
 $-4 \leq \frac{2x-4}{3} < 2$.

Solution:

$$\begin{array}{rcl}
 -4 \leq \frac{2x-4}{3} < 2 \\
 -12 \leq 2x - 4 < 6 \\
 -8 \leq 2x < 10 \\
 -4 \leq x < 5
 \end{array}$$

The solution set is $[-4, 5)$.

Example 1.2. Solve the linear inequality
 $-3x - 4 < 2$.

Solution:

$$\begin{array}{rcl}
 & -3x - 4 & < 2 \\
 \text{add } 4 & & -3x < 6 \\
 \text{divide by } -3 \text{ and switch} & & x > -2
 \end{array}$$

The solution set is $(-2, +\infty)$.

Example 1.4. Solve the compound linear inequality $-x + 4 > 2$ or $2x - 5 \geq -3$.

Solution:

$$\begin{array}{rcl}
 -x + 4 > 2 & \text{or} & 2x - 5 \geq -3 \\
 -x > -2 & & 2x \geq 2 \\
 x < 2 & \text{or} & x \geq 1
 \end{array}$$

That is $x \geq 1$ or $x < 2$. The solution set is $(-\infty, +\infty)$.

Example 1.6. Solve the compound linear inequality
 $-1 \leq \frac{-3x+4}{2} < 3$.

Solution:

$$\begin{array}{rcl}
 -1 \leq \frac{-3x+4}{2} < 3 \\
 -2 \leq -3x + 4 < 6 \\
 -6 \leq -3x < 2 \\
 2 \geq x > -\frac{2}{3}
 \end{array}$$

The solution set is $(-\frac{2}{3}, 2]$.

Lesson 1. Linear Inequalities

Practice 1.1. Solve the linear inequality. Write your answer in interval notation.

(1) $3x + 7 \leq 1$

(2) $2x - 3 > 1$

(1) $(-\infty, 2]$ (2) $(1, \infty)$

(1) $(-\infty, 2]$ (2) $(1, \infty)$

Practice 1.2. Solve the linear inequality. Write your answer in interval notation.

(1) $4x + 7 > 2x - 3$

(2) $3 - 2x \leq x - 6$

(1) $(-3, \infty)$ (2) $(-\infty, 2]$

(1) $(-3, \infty)$ (2) $(-\infty, 2]$

Practice 1.3. Solve the compound linear inequality. Write your answer in interval notation.

(1) $3x + 2 > -1$ and $2x - 7 \leq 1$.

(2) $4x - 7 < 5$ and $5x - 2 \geq 3$

(1) $(1, 3]$ (2) $(1, 3]$

(1) $(1, 3]$ (2) $(1, 3]$

Practice 1.4. Solve the compound linear inequality. Write your answer in interval notation.

(1) $-4 \leq 3x + 5 < 11$.

(2) $7 \geq 2x - 3 \geq -7$

(1) $[-2, 5]$ (2) $[-2, 5]$

(1) $[-2, 5]$ (2) $[-2, 5]$

Practice 1.5. Solve the compound linear inequality. Write your answer in interval notation.

(1) $3x - 5 > -2$ or $10 - 2x \leq 4$.

(2) $2x + 7 < 5$ or $3x - 8 \geq x - 2$

(1) $(-\infty, 3) \cup (1, \infty)$ (2) $(-\infty, 1) \cup (3, \infty)$

(1) $(-\infty, 3) \cup (1, \infty)$ (2) $(-\infty, 1) \cup (3, \infty)$

Practice 1.6. Solve the compound linear inequality. Write your answer in interval notation.

(1) $-2 \leq \frac{2x - 5}{3} < 3$.

(2) $-1 < \frac{3x + 7}{2} \leq 4$.

(1) $(\frac{1}{3}, 3]$ (2) $(\frac{1}{3}, 3]$

(1) $(\frac{1}{3}, 3]$ (2) $(\frac{1}{3}, 3]$



Lesson 2. Absolute Value Equations

Absolute Value Equations

The **absolute value** of a real number a , denoted by $|a|$, is the distance from 0 to a on the number line. In particular, $|a|$ is always greater than or equal to 0, that is $|a| \geq 0$. The absolute values have the following properties: $|-a| = |a|$, $|ab| = |a||b|$ and $|\frac{a}{b}| = \frac{|a|}{|b|}$.

Let X represent an algebraic expression.

If c is a **positive** number, then the equation $|X| = c$ is equivalent to

$$X = c \quad \text{or} \quad X = -c.$$

If c is a **negative** number, then the solution set of $|X| = c$ is **empty**. An **empty set** is denoted by \emptyset .

The equation $|X| = 0$ is equivalent to $X = 0$.

Example 2.1. Solve the equation $|2x - 3| = 7$.

Solution: The equation is equivalent to

$$\begin{aligned} 2x - 3 &= -7 & \text{or} & & 2x - 3 &= 7 \\ 2x &= -4 & & & 2x &= 10 \\ x &= -2 & \text{or} & & x &= 5 \end{aligned}$$

The solutions are $x = -2$ or $x = 5$. In set-builder notation, the solution set is $\{-2, 5\}$.

Example 2.3. Solve the equation

$$3|2x - 5| = 9.$$

Solution: Rewrite the equation into $|X| = c$ form.

$$|2x - 5| = 3$$

Solve the equation.

$$\begin{aligned} 2x - 5 &= -3 & \text{or} & & 2x - 5 &= 3 \\ 2x &= 2 & & & 2x &= 8 \\ x &= 1 & \text{or} & & x &= 4 \end{aligned}$$

The solutions are $x = 1$ or $x = 4$. In set-builder notation, the solution set is $\{1, 4\}$.

Example 2.5. Solve the equation

$$|3x - 2| = |x + 2|.$$

Solution: Because two numbers have the same absolute value only if they are the same or opposite to each other. Then the equation is equivalent to

$$\begin{aligned} 3x - 2 &= x + 2 & \text{or} & & 3x - 2 &= -(x + 2) \\ 2x &= 4 & & & 4x &= 0 \\ x &= 2 & \text{or} & & x &= 0 \end{aligned}$$

The solutions are $x = 2$ and $x = 0$. In set-builder notation, the solution set is $\{0, 2\}$.

Example 2.2. Solve the equation $|2x - 1| - 3 = 8$.

Solution: Rewrite the equation into $|X| = c$ form.

$$|2x - 1| = 11$$

Solve the equation.

$$\begin{aligned} 2x - 1 &= -11 & \text{or} & & 2x - 1 &= 11 \\ 2x &= -10 & & & 2x &= 12 \\ x &= -5 & \text{or} & & x &= 6 \end{aligned}$$

The solutions are $x = -5$ or $x = 6$. In set-builder notation, the solution set is $\{-5, 6\}$.

Example 2.4. Solve the equation

$$2|1 - 2x| - 3 = 7.$$

Solution: Rewrite the equation into $|X| = c$ form.

$$|2x - 1| = 5$$

Solve the equation.

$$\begin{aligned} 2x - 1 &= -5 & \text{or} & & 2x - 1 &= 5 \\ 2x &= -4 & & & 2x &= 6 \\ x &= -2 & \text{or} & & x &= 3 \end{aligned}$$

The solutions are $x = -2$ or $x = 3$. In set-builder notation, the solution set is $\{-2, 3\}$.

Example 2.6. Solve the equation

$$2|1 - x| = |2x + 10|.$$

Solution: This equation is similar to the one in Example 2.5 except that there is an extra number 2 outside. Since 2 is positive, we are free to move it inside. Moreover, because $|-X| = |X|$, the equation is equivalent to

$$|2x - 2| = |2x + 10|$$

$$\begin{aligned} 2x - 2 &= 2x + 10 & \text{or} & & 2x - 2 &= -(2x + 10) \\ -2 &= 10 & \text{or} & & 4x &= -8 \\ & & & & x &= -2 \end{aligned}$$

The original equation only has one solution $x = -2$. In set-builder notation, the solution set is $\{-2\}$.

Lesson 2. Absolute Value Equations

Practice 2.1. Find the solution set for the equation.

(1) $|2x - 1| = 5.$

(2) $\left| \frac{3x - 9}{2} \right| = 3.$

(1) $\{5, 1\}$ (2) $\{1, 2\}$

(1) $\{-2, 3\}$ (2) $\{1, 2\}$

Practice 2.2. Find the solution set for the equation.

(1) $|3x - 6| + 4 = 13.$

(2) $3|2x - 5| = 9.$

(1) $\{1, 4\}$ (2) $\{1, 2\}$

(1) $\{-1, 5\}$ (2) $\{1, 2\}$

Practice 2.3. Find the solution set for the equation.

(1) $|5x - 10| + 6 = 6.$

(2) $-3|3x - 11| = 5.$

(1) \emptyset (2) $\{2\}$

(1) $\{2\}$ (2) $\{2\}$

Practice 2.4. Find the solution set for the equation.

(1) $|5x - 12| = |3x - 4|.$

(2) $|x - 1| = -5|2 - x) - 1|.$

(1) $\{1, 2\}$ (2) $\{1, 2\}$

(1) $\{2, 4\}$ (2) $\{2, 4\}$



Lesson 3. Properties of Integral Exponents

Review on Properties of Exponents

An **exponent** is a number that we put on the upper right corner of a number, variable or expression, **called the base**, to tell us how many times we should multiply the base with itself. In algebra, we write

$$x^n = \underbrace{x \cdot x \cdots x}_{n \text{ factors of } x}.$$

Exponents have the following properties.

1. The product rule

$$x^m \cdot x^n = x^{m+n}.$$

Example:

$$2x^2 \cdot (-3x^3) = -6x^5.$$

2. The quotient rule (Suppose $x \neq 0$.)

$$\frac{x^m}{x^n} = \begin{cases} x^{m-n} & \text{if } m \text{ is greater than} \\ & \text{or equal to } n. \\ \frac{1}{x^{n-m}} & \text{if } m \text{ is less than } n. \end{cases}$$

Example:

$$\frac{15x^5}{5x^2} = 3x^3, \quad \frac{-3x^2}{6x^3} = -\frac{1}{2x}$$

3. The zero exponent rule (Suppose $x \neq 0$.)

$$x^0 = 1.$$

Example:

$$(-2)^0 = 1, \quad -x^0 = -1$$

4. The negative exponent rule (Suppose $x \neq 0$.) **Example:**

$$x^{-n} = \frac{1}{x^n} \quad \text{and} \quad \frac{1}{x^{-n}} = x^n.$$

$$(-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}, \quad \frac{x^{-2}}{x^{-3}} = \frac{x^3}{x^2} = x.$$

5. The power to a power rule:

$$(x^a)^b = x^{ab}.$$

Example:

$$(2^2)^3 = 2^6 = 64, \quad (x^2)^3 = x^6.$$

6. The product raised to a power rule:

$$(xy)^n = x^n y^n.$$

Example:

$$(-2x)^2 = (-2)^2 x^2 = 4x^2$$

$$(-a^2b)^3 = (-1)^3 (a^2)^3 b^3 = -a^6 b^3.$$

7. The quotient raised to a power rule (Suppose $y \neq 0$.) **Example:**

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

$$\left(\frac{x}{-2}\right)^3 = \frac{x^3}{(-2)^3} = -\frac{x^3}{8}, \quad \left(\frac{-a^2}{b^3}\right)^2 = \frac{(-a^2)^2}{(b^3)^2} = \frac{a^4}{b^6}.$$

Order of Basic Mathematical Operations

In mathematics, the order of operations reflects conventions about which procedure should be performed first. There are four levels (from the highest to the lowest):

Parenthesis; Exponent; Multiplication and Division; Addition and Subtraction.

Within the same level, the convention is to perform from the left to the right.

Example 3.1. Simplify $\left(\frac{2y^{-2}z^{-5}}{4x^{-3}y^4}\right)^{-4}$.

Solution:

$$\left(\frac{2y^{-2}z^{-5}}{4x^{-3}y^4}\right)^{-4} = \left(\frac{x^3}{2z^5y^6}\right)^{-4} = \left(\frac{2z^5y^6}{x^3}\right)^4 = \frac{2^4(z^5)^4(y^6)^4}{(x^3)^4} = \frac{16z^{20}y^{24}}{x^{12}}.$$

Lesson 3. Properties of Integral Exponents

Practice 3.1. Simplify. Write with positive exponents.

(1) $a^2a^3a^5$ (2) $\frac{-u^0v^{15}}{v^{16}}$ (3) $(-2)^{-3}$ (4) $(-x^{-1}(-y)^2)^3$

$\frac{x^x}{9^4}$ (7) I.E $\frac{8}{1}$ (8) I.E $\frac{a}{1}$ (2) I.E $0I^p$ (1) I.E

Practice 3.2. Simplify. Write with positive exponents.

(1) $(3a^2b^3c^2)(4abc^2)(2b^2c^3)$ (2) $\frac{4y^3z^0}{x^2y^2}$ (3) $(-2a^3b^2c^0)^3$ (4) $(2x^{-4})^3$

$\frac{z^1x}{8}$ (7) 2.E $9q_6^p8-$ (8) 2.E $\frac{z^x}{4^4}$ (2) 2.E $7^29^43^67^6$ (1) 2.E

Practice 3.3. Simplify. Write with positive exponents.

(1) $(-3a^2x^3)^{-2}$ (2) $\left(\frac{-x^0y^3}{2wz^2}\right)^3$ (3) $\frac{3^{-2}a^{-3}b^5}{x^{-3}y^{-4}}$ (4) $\left(\frac{6x^{-2}y^5}{2y^{-3}z^{-11}}\right)^{-3}$

$\frac{27^y z^4 z^3}{9^x}$ (7) 3.E $\frac{6^6}{4^5 x^3 x^5}$ (8) 3.E $\frac{6^2 \epsilon^m 8}{6^y}$ (2) 3.E $\frac{9^x + 6^6}{1}$ (1) 3.E



Lesson 4. Introduction to Functions

Functions are Relations

A **relation** is a set of ordered pairs. The set of all first components of the ordered pairs is called the **domain**. The set of all second components of the ordered pairs is called the **range**.

A **function** is a relation such that each element in the domain corresponds to *exactly one element in the range*.

Functions as Equations and Function Notation

For a function, we usually use the variable x to represent an element from the domain and call it the **independent variable**. The variable y is used to represent the value corresponding to x and is called the **dependent variable**. We say y is a function of x . When we consider several functions together, to distinguish them we named functions by a letter such as f , g , or F . The notation $f(x)$, read as “ f of x ” or “ f at x ”, represents the output of the function f when the input is x . To find the value of a function at a given number, we substitute the independent variable x by the given number and then evaluate the expression. We call the procedure evaluating a function.

Example 4.1. Find the indicated function value.

1. $f(-2)$, $f(x) = 2x + 1$ $f(-2) = 2 \cdot (-2) + 1 = -4 + 1 = -3$.

2. $g(2)$, $g(x) = 3x^2 - 10$ $g(-2) = 3 \cdot (2^2) - 10 = 3 \cdot 4 - 10 = 12 - 10 = 2$.

3. $h(a - t)$, $h(x) = 3x + 5$ $h(a - t) = 3 \cdot (a - t) + 5 = 3a - 3t + 5$.

Graphs of Functions and Graphs as Functions

The graph of a function is the graph of its ordered pairs.

A graph of ordered pairs (x, y) in the rectangular system defines y as a function of x if any vertical line crosses the graph at most once. This test is called the **vertical line test**.

The domain of a graph is the set of all inputs (x -coordinates). The range of a graph is the set of all outputs (y -coordinates).

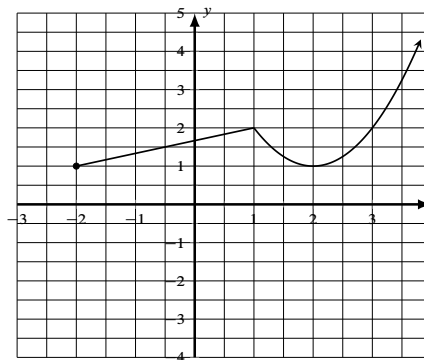
To find the domain of a graph, we look for the left and the right endpoints. To find the range of a graph, we look for the highest and the lowest positioned points.

Example 4.2. Use the graph on the right to answer the following questions.

(1) Determine whether the graph is a function and explain your answer.

(2) Find the domain of the graph (write the domain in interval notation).

(3) Find the range of the graph (write the range in interval notation).



Solution: (1) The graph is a function. Because every vertical line crosses the graph at most once.

(2) The graph has the left endpoint at $(-2, 1)$ and but no right endpoint. So the domain is $[-2, +\infty)$.

(3) The graph has two lowest positioned points $(-2, 1)$ and $(2, 1)$ and but no highest positioned point. So the range is $[1, +\infty)$.

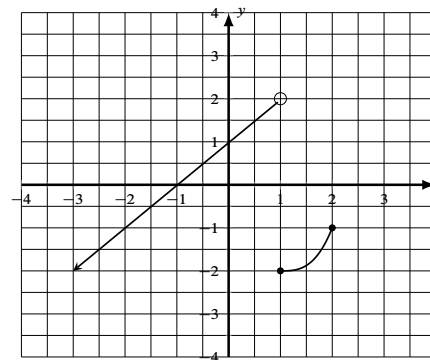
Lesson 4. Introduction to Functions

Practice 4.1. Find the indicated function values for the functions $f(x) = -x^2 + x - 1$ and $g(x) = 2x - 1$. Simplify your answer.

- (1) $f(2)$ (2) $f(-x)$ (3) $g(-1)$ (4) $g(f(1))$

4.1 (1) $f(2) = -3$ 4.1 (2) $f(-x) = -x^2 - x - 1$ 4.1 (3) $g(-1) = -3$ 4.1 (4) $g(f(1)) = -3$

Practice 4.2. Use the graph on the right to answer the following questions.



- (1) Determine whether the graph is a function and explain your answer.
- (2) Find the domain of the graph (write the domain in interval notation).
- (3) Find the range of the graph (write the range in interval notation).

4.2 (1) Yes 4.2 (2) $(-\infty, 2]$ 4.2 (3) $(-\infty, 2)$

Practice 4.3. Find the corresponding x -coordinate of a point on the graph in Practice 4.2 whose y -coordinate is 1.

4.3 $x = 0$



Lesson 5. Linear functions

The Slope-Intercept Form Equation

The slope of a line measures the steepness, in other words, “rise” over “run”, or rate of change of the line. Using the rectangular coordinate system, the **slope** m of a line is defined as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}},$$

where (x_1, y_1) and (x_2, y_2) are any two distinct points on the line. If the line intersects the y -axis at the point $(0, b)$, then a point (x, y) is on the line if and only if

$$y = mx + b.$$

This equation is called the **slope-intercept form** of the line.

Linear Function

A **linear function** is a function whose graph is a line. A linear functions can be written as

$$f(x) = mx + b,$$

where m is the slope and $(0, b)$ is the y -intercept.

If a function $f(x)$ is a linear function, then $b = f(0)$ and the slope

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{change in output}}{\text{change in input}}.$$

How to Write an Equation for a Linear Function?

Example 5.1. If $f(x)$ is a linear function, with $f(2) = 5$, and $f(-1) = 2$, find an equation for the function.

Step 1. Find the slope m : $m = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{5 - 2}{2 - (-1)} = \frac{3}{3} = 1$.

Step 2. Plug $x = 2$ into the slope-intercept form $f(x) = 1x + b$ and then solve for b :

$$5 = f(2) = 2 + b$$

$$5 + (-2) = 2 + (-2) + b$$

$$3 = b$$

Step 3. The slope-intercept form equation of this function is $f(x) = x + 3$.

Sketch the Graph of a Linear Function via Plotting Points

Example 5.2. Sketch the graph of the linear function $f(x) = -\frac{1}{2}x + 1$.

Method 1: Get points by evaluating $f(x)$.

Step 1. Choose two or more input values, e.g. $x = 0$ and $x = 2$.

Step 2. Evaluate $f(x)$: $f(0) = 1$ and $f(2) = 0$.

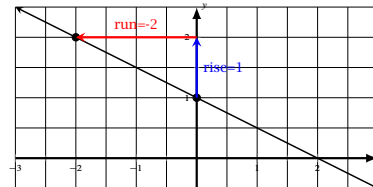
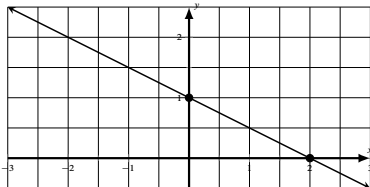
Step 3. Plot the points $(0, 1)$ and $(2, 0)$ and draw a line through them.

Method 2: Get points by raise and run.

Step 1. Plot the y -intercept $(0, f(0)) = (0, 1)$.

Step 2. Use $\frac{\text{rise}}{\text{run}} = -\frac{1}{2}$ to get one or more points, e.g. we will get $(-2, 2)$ by taking rise = 1 and run = -2, i.e. move up 1 unit, then move to the right 2 units.

Step 3. Plot the points $(0, 1)$ and $(-2, 2)$ and draw a line through them.



Lesson 5. Linear functions

Practice 5.1. Finding the slope of the line passing through $(3, 5)$ and $(-1, 1)$.

5.1 $m = 1$

Practice 5.2. Find the slope-intercept form equation of the line passing through $(6, 3)$ and $(2, 5)$.

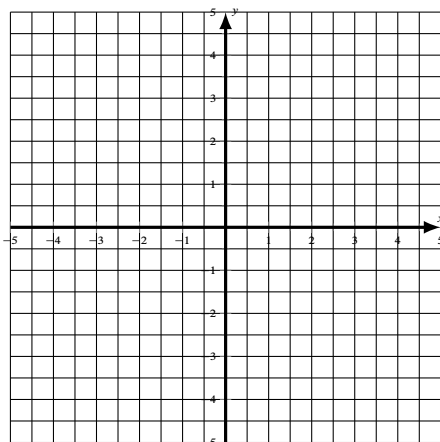
5.2 $y = -\frac{1}{2}x + 9$

Practice 5.3. Determine if the linear functions $f(x)$ and $h(x)$ with the following values $f(-2) = -4$, $f(0) = h(0) = 2$ and $h(2) = 8$ define the same function. Explain your answer.

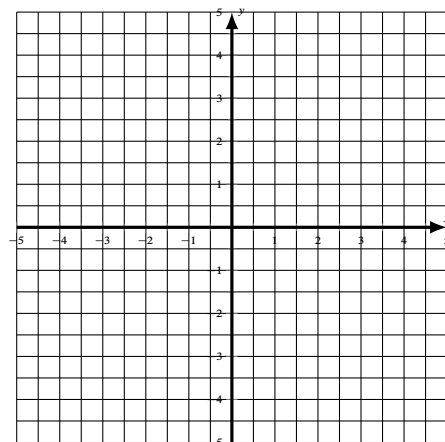
5.3 Yes. Because $f(x) = 3x + 2$ and $g(x) = 3x + 2$.

Practice 5.4. Graph the function.

(1) $f(x) = -x + 1$



(2) $f(x) = \frac{1}{2}x - 1$



Lesson 6. Perpendicular and Parallel Lines

Horizontal and Vertical Lines

A **horizontal line** is defined by an equation $y = b$. The slope of a horizontal line is simply zero. A **vertical line** is defined by an equation $x = a$. The slope of a vertical line is **undefined**.

A vertical line gives an example that a graph is not a function of x . Indeed, the vertical line test fails for a vertical line.

Get the Explicit Function from an Equation

When studying functions, we prefer a clearly expressed function rule. For example, in $f(x) = -\frac{2}{3}x + 1$, the expression $-\frac{2}{3}x + 1$ clearly tells us how to produce outputs. For a function f defined by an equation, for instance, $2x + 3y = 3$, to find the function rule (that is an expression), we simply solve the given equation for y .

$$\begin{aligned}2x + 3y &= 3 \\3y &= -2x + 3 \\y &= -\frac{2}{3}x + 1.\end{aligned}$$

Now, we get $f(x) = -\frac{2}{3}x + 1$.

Point-Slope Form Equation of a Line

When the slope m of a line and a point (x_0, y_0) on the line are given, we can write down an equation for the line in the following form, called the **point-slope form**:

$$y - y_0 = m(x - x_0).$$

This equation comes essentially from the slope formula

$$m = \frac{y - y_0}{x - x_0}.$$

Perpendicular and Parallel Lines

Any two vertical lines are parallel. Two non-vertical lines are **parallel** if and only if they **have the same slope**.

A line that is parallel to the line $y = mx + a$ has an equation $y = mx + b$, where $a \neq b$.

Horizontal lines are perpendicular to vertical lines. Two non-vertical lines are **perpendicular** if and only if **the product of their slopes is -1** .

A line that is perpendicular to the line $y = mx + a$ has an equation $y = -\frac{1}{m}x + b$.

Find Equations for Perpendicular or Parallel Lines

Example 6.1. Find an equation of the line that is parallel to the line $4x + 2y = 1$ and passes through the point $(-3, 1)$.

Step 1. Find the slope m of the original line from the slope-intercept form equation by solving for y . $y = -2x + \frac{1}{2}$. So $m = -2$.

Step 2. Find the slope m_{\parallel} of the parallel line.

$$m_{\parallel} = m = -2.$$

Step 3. Use the point-slope form.

$$\begin{aligned}y - 1 &= -2(x + 3) \\y &= -2x - 5.\end{aligned}$$

Example 6.2. Find an equation of the line that is perpendicular to the line $4x - 2y = 1$ and passes through the point $(-2, 3)$.

Step 1. Find the slope m of the original line from the slope-intercept form equation by solving for y . $y = 2x - \frac{1}{2}$. So $m = 2$.

Step 2. Find the slope m_{\perp} of the perpendicular line.

$$m_{\perp} = -\frac{1}{m} = -\frac{1}{2}.$$

Step 3. Use the point-slope form.

$$\begin{aligned}y - 3 &= -\frac{1}{2}(x + 2) \\y &= -\frac{1}{2}x + 2.\end{aligned}$$

Lesson 6. Perpendicular and Parallel Lines

Practice 6.1. Find the point-slope form equation of the line with slope 5 that passes through $(-2, 1)$.

$$(y - 1) = 5(x + 2)$$

Practice 6.2. Find the point-slope form equation of the line passing through $(3, -2)$ and $(1, 4)$.

$$(y - 4) = -\frac{3}{2}(x - 1)$$

Practice 6.3. Find the slopes of the lines that are parallel and perpendicular to the line $2x - 5y = -3$.

$$m_{\text{parallel}} = \frac{2}{5}, \quad m_{\text{perpendicular}} = \frac{5}{2}$$

Practice 6.4. Find the point-slope form and then the slope-intercept form equations of the line parallel to $3x - y = 4$ and passing through the point $(2, -3)$.

$$y - (-3) = 3(x - 2) \quad y = 3x - 9$$

Practice 6.5. Find the slope-intercept form equation of the line that is perpendicular to $4y - 2x + 3 = 0$ and passing through the point $(2, -5)$

$$y + 5 = -\frac{1}{2}(x - 2)$$



Lesson 7. Systems of Linear Equations

The Substitution Method and the Addition Method (the Elimination Method)

A **system of linear equations** of two variables consists of two equations. A **solution of a system** of linear equations of two variables is an ordered pair that satisfies both equations.

Substitution Method

Example 7.1. Solve the system of linear equations using the substitution method.

$$\begin{aligned} x + y &= 3 & (1) \\ 2x + y &= 4 & (2) \end{aligned}$$

Solution:

Step 1. Solve one variable from one equation. For example, one may solve y from equation (1).

$$y = 3 - x.$$

Step 2. Plug $y = 3 - x$ into the second equation and solve for x .

$$\begin{aligned} 2x + (3 - x) &= 4 \\ x + 3 &= 4 \\ x &= 1 \end{aligned}$$

Step 3. Plug the solution $x = 1$ into the equation in Step 1 to solve for y .

$$y = 3 - x = 3 - 1 = 2$$

Step 4. Check the proposed solution. Plug (1, 2) into the first equation:

$$1 + 2 = 3.$$

Plug (1, 2) into the second equation:

$$2 \cdot 1 + 2 = 4.$$

Elimination Method

Example 7.2. Solve the system of linear equations using the addition method.

$$\begin{aligned} 5x + 2y &= 7 & (1) \\ 3x - y &= 13 & (2) \end{aligned}$$

Solution:

Step 1. Eliminate one variable and solve for the other. For example, one may choose to eliminate y . In order to eliminate y , we *add the opposite*. We multiply both sides of the second equation by 2 to get the opposite $-2y$.

$$\begin{aligned} 2(3x) - 2y &= 2(13) \\ 6x - 2y &= 26 & (3) \end{aligned}$$

Adding equations (1) and (3) will eliminate y .

$$\begin{array}{r} 5x + 2y = 7 \\ + \quad 6x - 2y = 26 \\ \hline 11x + 0 = 33 \\ x = 3 \end{array}$$

Step 2. Find the missing variable by plugging $x = 3$ into the first equation and solve for y .

$$\begin{aligned} 5 \cdot 3 + 2y &= 7 \\ 15 + 2y &= 7 \\ 2y &= -8 \\ y &= -4 \end{aligned}$$

Step 3. Check the proposed solution. Plug (3, -4) into the first equation:

$$5 \cdot 3 + 2 \cdot (-4) = 15 - 8 = 7.$$

Plug (3, -4) into the second equation:

$$3 \cdot 3 - (-4) = 9 + 4 = 13.$$

Note: A linear system may have **one solution**, **no solution** or **infinitely many solutions**.

If the lines defined by equations in the linear system have the same slope but different y -intercepts or the elimination method ends up with $0 = c$, where c is a nonzero constant, then the system has no solution.

If the lines defined by equations in the linear system have the same slope and the same y -intercept or the elimination method ends up with $0 = 0$, then the system has infinitely many solutions. In this case, we say that the system is **dependent** and a solution can be expressed in the form $(x, f(x)) = (x, mx + b)$.

Lesson 7. Systems of Linear Equations

Practice 7.1. Solve.

$$\begin{aligned}2x - y &= 8 \\ -3x - 5y &= 1\end{aligned}$$

Practice 7.2. Solve.

$$\begin{aligned}x + 4y &= 10 \\ 3x - 2y &= -12\end{aligned}$$

Practice 7.3. Solve.

$$\begin{aligned}-x - 5y &= 29 \\ 7x + 3y &= -43\end{aligned}$$

Practice 7.4. Solve.

$$\begin{aligned}2x + 15y &= 9 \\ x - 18y &= -21\end{aligned}$$

Practice 7.5. Solve.

$$\begin{aligned}2x + 7y &= 5 \\ 3x + 2y &= 16\end{aligned}$$

Practice 7.6. Solve.

$$\begin{aligned}4x + 3y &= -10 \\ -2x + 5y &= 18\end{aligned}$$

Practice 7.7. Solve.

$$\begin{aligned}3x + 2y &= 6 \\ 6x + 4y &= 16\end{aligned}$$

Practice 7.8. Solve.

$$\begin{aligned}2x - 3y &= -6 \\ -4x + 6y &= 12\end{aligned}$$

$$\begin{aligned}7.8 & (x, \frac{3}{2}x + 1) \\ 7.4 & (-3, 1)\end{aligned}$$

$$\begin{aligned}7.7 & \text{no solution} \\ 7.3 & (-4, -5)\end{aligned}$$

$$\begin{aligned}7.6 & (-4, 2) \\ 7.2 & (-2, 3)\end{aligned}$$

$$\begin{aligned}7.5 & (6, -1) \\ 7.1 & (3, -2)\end{aligned}$$



Lesson 8. Factoring Review

Factor by Removing the GCF

The **greatest common factor (GCF)** of two terms is a polynomial with the **greatest coefficient** and of the **highest possible degree** that divides each term.

To **factor a polynomial** is to **express the polynomial as a product** of polynomials of lower degrees. The first and the easiest step is to factor out the GCF of all terms.

Example 8.1. Factor $4x^3y - 8x^2y^2 + 12x^3y^3$.

Solution: Step 1. Find the GCF of all terms. The GCF of $4x^3y$, $-8x^2y^2$ and $12x^4y^3$ is $4x^2y$.

Step 2. Write each term as the product of the GCF and the remaining factor.

$$4x^3y = (4x^2y) \cdot x, -8x^2y^2 = (4x^2y) \cdot (-2y), \text{ and } 12x^4y^3 = 3x^2y^2 \cdot 4x^2y.$$

Step 3. Factor out the GCF from each term.

$$4x^3y - 8x^2y^2 + 12x^3y^3 = 4x^2y \cdot (x - 2y + 3x^2y^2).$$

Factor a Four-term Polynomial by Grouping

For a four-term polynomial, in general, we will group them into two groups and factor out the GCF for each group and then factor further.

Example 8.2. Factor $2x^2 - 6xy + xz - 3yz$.

Solution:

Step 1. Group the first two terms and the last two terms.

$$\begin{aligned} &2x^2 - 6xy + xz - 3yz \\ &= (2x^2 - 6xy) + (xz - 3yz) \end{aligned}$$

Step 2. Factor out the GCF from each group.

$$= 2x(x - 3y) + z(x - 3y)$$

Step 3. Factor out the binomial GCF.

$$= (x - 3y)(2x + z).$$

Example 8.3. Factor $ax + 4b - 2a - 2bx$.

Solution:

Step 1. Group the first term with the third term and group the second term and the last term.

$$\begin{aligned} &ax + 4b - 2a - 2bx \\ &= (ax - 2a) + (-2bx + 4b) \end{aligned}$$

Step 2. Factor out the GCF from each group.

$$= a(x - 2) + (-2b)(x - 2)$$

Step 3. Factor out the binomial GCF.

$$= (x - 2)(a - 2b).$$

Factor Binomials in Special Forms

Difference of squares:

$$a^2 - b^2 = (a - b)(a + b).$$

Example 8.4. Factor $25x^2 - 16$.

Solution:

Step 1. Recognize the binomial as a difference of squares.

$$\begin{aligned} &25x^2 - 16 \\ &= (5x)^2 - 4^2 \end{aligned}$$

Step 2. Apply the formula.

$$= (5x - 4)(5x + 4).$$

Sum of Cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

Example 8.5. Factor $x^3 + 27$.

Solution:

Step 1. Recognize the binomial as a sum of cubes.

$$\begin{aligned} &x^3 + 27 \\ &= x^3 + 3^3 \end{aligned}$$

Step 2. Apply the formula.

$$\begin{aligned} &= (x + 3)(x^2 - 3x + 3^2) \\ &= (x + 3)(x^2 - 3x + 9). \end{aligned}$$

Difference of cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Example 8.6. Factor $125x^3 - 8$.

Solution:

Step 1. Recognize the binomial as a difference of cubes.

$$\begin{aligned} &125x^3 - 8 \\ &= (5x)^3 - 2^3 \end{aligned}$$

Step 2. Apply the formula.

$$\begin{aligned} &= (5x - 2)((5x)^2 + 2(5x) + 2^2) \\ &= (5x - 2)(25x^2 + 10x + 4). \end{aligned}$$

Lesson 8. Factoring Review

Practice 8.1. Factor out the GCF.

(1) $18x^2y^2 - 12xy^3 - 6x^3y^4$

(2) $5x(x - 7) + 3y(x - 7)$

$(18x^2y^2 - 12xy^3 - 6x^3y^4)$ (1) 1.8

$(5x(x - 7) + 3y(x - 7))$ (1) 1.8

Practice 8.2. Factor by grouping.

(1) $12xy - 10y + 18x - 15$

(2) $12ac - 18bc - 10ad + 15bd$

(3) $5ax - 4bx - 5ay + 4by$

$(12xy - 10y + 18x - 15)$ (1) 2.8

$(12ac - 18bc - 10ad + 15bd)$ (2) 2.8

$(5ax - 4bx - 5ay + 4by)$ (1) 2.8

Practice 8.3. Factor completely.

(1) $25x^2 - 4$

(2) $8x^3 - 27$

(3) $125y^3 + 1$

$(25x^2 - 4)$ (1) 3.8

$(8x^3 - 27)$ (2) 3.8

$(125y^3 + 1)$ (1) 3.8

Practice 8.4. Factor completely.

(1) $27x^4 + xy^3$

(2) $16xy^3 - 2x^4$

(3) $x^3 + 3x^2 - 4x - 12$

$(27x^4 + xy^3)$ (1) 4.4 (1) 4.4 (1) 4.4



Lesson 9. Solve Quadratic Equations by Factoring

Factor Trinomials

If a trinomial $Ax^2 + Bx + C$ can be factored, then it can be expressed as a product of two binomials:

$$Ax^2 + Bx + C = (mx + n)(px + q).$$

By simplify the product using the FOIL method and comparing coefficients, we observe that

$$A = \underbrace{mn}_F \quad B = \underbrace{mq}_O + \underbrace{np}_I \quad C = \underbrace{nq}_L$$

The observation suggests the following strategy called the method of undetermined coefficients.

Example 9.1. Factor $x^2 + 6x + 8$.

Solution:

Step 1. Factor $A = 1$:

$$1 = 1 \cdot 1.$$

Step 2. Factor $C = 8$:

$$8 = 1 \cdot 8 = 2 \cdot 4.$$

Step 3. Choose a proper combination of pairs of factors and check if the sum of cross product equals $B = 6$:

$$1 \cdot 4 + 1 \cdot 2 = 6.$$

This step can be checked easily using the following diagram.

$$\begin{array}{r}
 A = 1 = 1 \cdot 1 \quad C = 8 = 2 \cdot 4 \\
 \begin{array}{ccc}
 1 & & 2 \\
 1 & & 4 \\
 \hline
 1 \cdot 2 & + & 1 \cdot 4 \\
 & & = 6 = B
 \end{array}
 \end{array}$$

Step 4. Factor the trinomial

$$x^2 + 6x + 8 = (x + 2)(x + 4).$$

Example 9.2. Factor $2x^2 + 5x - 3$.

Solution:

Step 1. Factor $A = 2$:

$$1 = 1 \cdot 2.$$

Step 2. Factor $C = -3$:

$$-3 = 1 \cdot (-3) = (-1) \cdot 3.$$

Step 3. Choose a proper combination of pairs of factors and if the sum of cross products equals $B = 5$:

$$2 \cdot 3 + 1 \cdot (-1) = 5.$$

This step can be checked easily using the following diagram.

$$\begin{array}{r}
 A = 2 = 1 \cdot 2 \quad C = -3 = 3 \cdot (-1) \\
 \begin{array}{ccc}
 1 & & 3 \\
 2 & & -1 \\
 \hline
 2 \cdot 3 & + & 1 \cdot (-1) \\
 & & = 5 = B
 \end{array}
 \end{array}$$

Step 4. Factor the trinomial

$$2x^2 + 5x - 3 = (x + 3)(2x - 1).$$

The following example shows how to factor a certain higher degree polynomial by substitution.

Example 9.3. Factor the trinomial completely.

$$4x^4 - x^2 - 3$$

Solution:

Step 1. Let $x^2 = y$. Then $4x^4 - x^2 - 3 = 4y^2 - y - 3$.

Step 2. Factor the trinomial in y : $4y^2 - y - 3 = (4y + 3)(y - 1)$.

Step 3. Replace y by x^2 and factor further.

$$\begin{aligned}
 4x^4 - x^2 - 3 &= 4y^2 - y - 3 \\
 &= (4y + 3)(y - 1) \\
 &= (4x^2 + 3)(x^2 - 1) \\
 &= (4x^2 + 3)(x - 1)(x + 1).
 \end{aligned}$$

Lesson 9. Solve Quadratic Equations by Factoring

Solving a Quadratic Equation by Factoring

A **quadratic equation** is a polynomial equation of degree 2, for example, $2x^2 + 5x - 3 = 0$. The **standard form** of a quadratic equation is

$$ax^2 + bx + c = 0,$$

where a , b and c are numbers, and $a \neq 0$.

To solve a quadratic equation, we may first factor the polynomial and then apply the **zero product property**:

$$A \cdot B = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0.$$

Example 9.4. Solve the equation
 $2x^2 + 5x = 3$.

Solution:

Step 1. Rewrite the equation into “**Expression=0**” form and factor.

$$\begin{aligned} 2x^2 + 5x &= 3 \\ 2x^2 + 5x - 3 &= 0 \\ (2x - 1)(x + 3) &= 0 \end{aligned}$$

Step 2. Apply the zero product property.

$$2x - 1 = 0 \quad \text{or} \quad x + 3 = 0.$$

Step 3. Solve each equation.

$$\begin{aligned} 2x - 1 &= 0 & \text{or} & & x + 3 &= 0 \\ 2x &= 1 & & & x &= -3 \\ x &= \frac{1}{2} & \text{or} & & x &= -3 \end{aligned}$$

Step 4. The solution set is $\{-3, \frac{1}{2}\}$.

Example 9.5. Solve the equation
 $(x - 2)(x + 3) = -4$.

Solution:

Step 1. Rewrite the equation into “**Expression=0**” form and factor.

$$\begin{aligned} (x - 2)(x + 3) &= -4 \\ x^2 + x - 6 &= -4 \\ x^2 + x - 2 &= 0 \\ (x - 1)(x + 2) &= 0 \end{aligned}$$

Step 2. Apply the zero product property.

$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0.$$

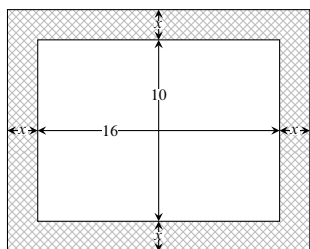
Step 3. Solve each equation.

$$\begin{aligned} x - 1 &= 0 & \text{or} & & x + 2 &= 0 \\ x &= 1 & \text{or} & & x &= -2 \end{aligned}$$

Step 4. The solution set is $\{-2, 1\}$.

Applications of Quadratic Equations

Example 9.6. A rectangular garden is surrounded by a path of uniform width. If the dimension of the garden is 10 meters by 16 meters and the total area is 216 square meters, determine the width of the path.



Solution:

Step 1. Suppose that the width of the frame is x meters. Translate given information into ex-

pressions in x and build an equation.

Total Width: $2x+10$ Total Length: $2x+16$

Width \times Length = Total Area:

$$(2x + 10)(2x + 16) = 216.$$

Step 2. Solve the equation.

$$\begin{aligned} (2x + 10)(2x + 16) &= 216 \\ 4x^2 + 52x + 160 &= 216 \\ 4x^2 + 52x - 56 &= 0 \\ x^2 + 13x - 14 &= 0 \\ (x + 14)(x - 1) &= 0 \end{aligned}$$

$$x = -14 \quad \text{or} \quad x = 1$$

Step 3. So the width of the path is 1 meter.

Lesson 9. Solve Quadratic Equations by Factoring

Practice 9.1. Factor the trinomial.

- (1) $x^2 + 4x + 3$ (2) $x^2 + 6x - 7$ (3) $x^2 - 3x - 10$ (4) $x^2 - 5x + 6$

(5) $(x - 2)(x + 1)$ (6) $(x + 3)(x - 4)$ (7) $(x - 1)(x + 2)$ (8) $(x + 5)(x - 6)$ (9) $(x - 7)(x + 8)$ (10) $(x + 9)(x - 10)$

Practice 9.2. Factor the trinomial.

- (1) $5x^2 + 7x + 2$ (2) $2x^2 + 5x - 12$ (3) $3x^2 - 10x - 8$ (4) $4x^2 - 12x + 5$

(5) $(5 - x)(1 - 2x)$ (6) $(4 + x)(3 - x)$ (7) $(3 - x)(2 + x)$ (8) $(2 + x)(1 + 5x)$ (9) $(1 - 2x)(4 + x)$ (10) $(2 + x)(5 + 1)$

Practice 9.3. Solve the equation by factoring.

- (1) $x^2 - 3x + 2 = 0$ (2) $2x^2 - 3x = 5$ (3) $(x - 1)(x + 3) = 5$

(4) $\{2, 4\}$ (5) $\{1, 2\}$ (6) $\{-1, \frac{2}{3}\}$ (7) $\{1, 2\}$ (8) $\{1, 2\}$

Practice 9.4. A paint measuring 3 inches by 4 inches is surrounded by a frame of uniform width. If the combined area of the paint and the frame is 30 square inches, determine the width of the frame.

9.4 1 inches.

Practice 9.5. A rectangle whose length is 2 meters longer than its width has an area 8 square meters. Find the width and the length of the rectangle.

9.5 width: 2 meters length: 4 meters.

Practice 9.6. The product of two consecutive negative odd numbers is 35. Find the numbers.

9.6 The numbers are -7 and -5.



Lesson 10. Multiply or Divide Rational Expressions

What is a Rational Expression and How to Simplify It?

A **rational expression** is a fraction $\frac{p}{q}$, where **the numerator** p and **the denominator** q are both polynomials and the degree of q is nonzero.

A rational expression is **simplified** if the numerator and the denominator have no common factor other than 1.

If p , q and c are polynomials with $q \neq 0$ and $c \neq 0$, then $\frac{p \cdot c}{q \cdot c} = \frac{p}{q}$.

Example 10.1. Simplify $\frac{x^2 + 4x + 3}{x^2 + 3x + 2}$.

Solution:

Step 1. Factor both the top and the bottom.

$$\frac{x^2 + 4x + 3}{x^2 + 3x + 2} = \frac{(x + 1)(x + 3)}{(x + 1)(x + 2)}$$

Step 2. Divide out common factors.

$$\frac{(x + 1)(x + 3)}{(x + 1)(x + 2)} = \frac{x + 3}{x + 2}$$

Example 10.2. Simplify $\frac{2x^2 - x - 3}{2x^2 - 3x - 5}$.

Solution:

Step 1. Factor both the top and the bottom.

$$\frac{2x^2 - x - 3}{2x^2 - 3x - 5} = \frac{(x + 1)(2x - 3)}{(x + 1)(2x - 5)}$$

Step 2. Divide out common factors.

$$\frac{(x + 1)(2x - 3)}{(x + 1)(2x - 5)} = \frac{2x - 3}{2x - 5}$$

How to Multiply Rational Expressions?

If p , q , s , t are polynomials with $q \neq 0$ and $t \neq 0$, then

$$\frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt}$$

Example 10.3. Multiply and then simplify.

$$\frac{3x^2}{x^2 + x - 6} \cdot \frac{x^2 - 4}{6x}$$

Solution:

Step 1. Factor numerators and denominators.

$$\frac{3x^2}{x^2 + x - 6} \cdot \frac{x^2 - 4}{6x} = \frac{3 \cdot x \cdot x}{(x - 2)(x + 3)} \cdot \frac{(x - 2)(x + 2)}{2 \cdot 3 \cdot x}$$

Step 2. Multiply and simplify.

$$\frac{\cancel{3} \cdot \cancel{x} \cdot x \cdot \cancel{(x - 2)}(x + 2)}{\cancel{(x - 2)}(x + 3) \cdot 2 \cdot \cancel{3} \cdot \cancel{x}} = \frac{x(x + 2)}{2(x + 3)}$$

Example 10.4. Multiply and then simplify.

$$\frac{3x^2 - 8x - 3}{x^2 + 8x + 16} \cdot \frac{x^2 - 16}{5x^2 - 14x - 3}$$

Solution:

$$\begin{aligned} &= \frac{(3x + 1)(x - 3)}{(x + 4)(x + 4)} \cdot \frac{(x + 4)(x - 4)}{(5x + 1)(x - 3)} \\ &= \frac{(3x + 1)\cancel{(x - 3)}\cancel{(x + 4)}(x - 4)}{\cancel{(x + 4)}(x + 4)(5x + 1)\cancel{(x - 3)}} \\ &= \frac{(3x + 1)(x - 4)}{(x + 4)(5x + 1)} \end{aligned}$$

How to Divide Rational Expressions?

If p , q , s , t are polynomials where $q \neq 0$, $s \neq 0$ and $t \neq 0$, then

$$\frac{p}{q} \div \frac{s}{t} = \frac{p}{q} \cdot \frac{t}{s} = \frac{pt}{qs}$$

Example 10.5. Divide and then simplify.

$$\frac{2x + 6}{x^2 - 6x - 7} \div \frac{6x - 2}{2x^2 - x - 3}$$

Solution:

Step 1. Rewrite the division as a multiplication.

$$\frac{2x + 6}{x^2 - 6x - 7} \div \frac{6x - 2}{2x^2 - x - 3} = \frac{2x + 6}{x^2 - 6x - 7} \cdot \frac{2x^2 - x - 3}{6x - 2}$$

Step 2. Factor and simplify.

$$\frac{2x + 6}{x^2 - 6x - 7} \cdot \frac{2x^2 - x - 3}{6x - 2} = \frac{2(x + 3)}{(x + 1)(x - 7)} \cdot \frac{(x + 1)(2x - 3)}{2(3x - 1)} = \frac{2(x + 3)(x + 1)(2x - 3)}{2(x + 1)(x - 7)(3x - 1)} = \frac{(x + 3)(2x - 3)}{(x - 7)(3x - 1)}$$

Lesson 10. Multiply or Divide Rational Expressions

Practice 10.1. Simplify.

(1) $\frac{3x^2 - x - 4}{x + 1}$

(2) $\frac{2x^2 - x - 3}{2x^2 + 3x + 1}$

(3) $\frac{x^2 - 9}{3x^2 - 8x - 3}$

$\frac{1 + x}{x + 3}$ (1) 10.1

$\frac{1 + x}{2x - 3}$ (2) 10.1

$4 - x$ (1) 10.1

Practice 10.2. Multiply and simplify.

(1) $\frac{x + 5}{x + 4} \cdot \frac{x^2 + 3x - 4}{x^2 - 25}$

(2) $\frac{3x^2 - 2x}{x + 2} \cdot \frac{3x^2 - 4x - 4}{9x^2 - 4}$

(3) $\frac{6y - 2}{3 - y} \cdot \frac{y^2 - 6y + 9}{3y^2 - y}$

$\frac{y}{(y - 3)^2}$ (3) 10.2

$\frac{z + x}{(z - x)^2}$ (2) 10.2

$\frac{5 - x}{x - 1}$ (1) 10.2

Practice 10.3. Divide and simplify.

(1) $\frac{9x^2 - 49}{6} \div \frac{3x^2 - x - 14}{2x + 4}$

(2) $\frac{x^2 + 3x - 10}{2x - 2} \div \frac{x^2 - 5x + 6}{x^2 - 4x + 3}$

(3) $\frac{y - x}{xy} \div \frac{x^2 - y^2}{y^2}$

$\frac{(y + x)x}{y}$ (3) 10.3

$\frac{z}{x + 5}$ (2) 10.3

$\frac{3}{3x + 7}$ (1) 10.3



Lesson 11. Add or Subtract Rational Expressions

Add or Subtract Rational Expressions with the Same Denominator

If P , Q and R are polynomials with $R \neq 0$, then

$$\frac{P}{R} + \frac{Q}{R} = \frac{P + Q}{R} \quad \text{and} \quad \frac{P}{R} - \frac{Q}{R} = \frac{P - Q}{R}.$$

Example 11.1. Add and simplify $\frac{x^2 + 4}{x^2 + 3x + 2} + \frac{5x}{x^2 + 3x + 2}$.

Solution:

Step 1. Determine if the rational expressions have the same denominator. If so, the new numerator will be the sum/difference of the numerators.

$$\frac{x^2 + 4}{x^2 + 3x + 2} + \frac{5x}{x^2 + 3x + 2} = \frac{x^2 + 5x + 4}{x^2 + 3x + 2}.$$

Step 2. Simplify the resulting rational expression.

$$\frac{x^2 + 5x + 4}{x^2 + 3x + 2} = \frac{(x + 1)(x + 4)}{(x + 1)(x + 2)} = \frac{x + 4}{x + 2}.$$

Add or Subtract Rational Expressions with Different Denominators

To add or subtract rational expressions with different denominators, we need to rewrite the rational expressions to equivalent rational expressions with the same denominator, say the LCD.

Step 1. Find the least common denominator (LCD).

Step 2. Rewrite each rational expression into an equivalent rational expression with the LCD as the new denominator.

Step 3. Add or subtract the resulting rational expressions and simplify.

Example 11.2. Find the LCD of $\frac{3}{x^2 - x - 6}$ and $\frac{6}{x^2 - 4}$.

Solution:

Step 1. Factor each denominator.

$$x^2 - x - 6 = (x + 2)(x - 3) \quad x^2 - 4 = (x - 2)(x + 2)$$

Step 2. List the factors of the first denominator and add unlisted factors of the second factor to get the final list.

First list	$(x + 2)$	$(x - 3)$	
Second list	$(x + 2)$		$(x - 2)$
Final list	$(x + 2)$	$(x - 3)$	$(x - 2)$

Step 3. The LCD is the product of factors in the final list.

$$(x + 2)(x - 3)(x - 2).$$

Example 11.3. Subtract and simplify

$$\frac{x - 3}{x^2 - 2x - 8} - \frac{1}{x^2 - 4}$$

Step 1. Find the LCD.

$$x^2 - 2x - 8 = (x + 2)(x - 4)$$

$$x^2 - 4 = (x - 2)(x + 2)$$

First list	$(x + 2)$		$(x - 4)$
Second list	$(x + 2)$	$(x - 2)$	
Final list	$(x + 2)$	$(x - 2)$	$(x - 4)$

The LCD is $(x + 2)(x - 2)(x - 4)$.

Step 2. Rewrite each rational expression into an equivalent expression with the LCD as the new denominator.

$$\frac{x - 3}{x^2 - 2x - 8} - \frac{1}{x^2 - 4} = \frac{(x - 3)(x - 2)}{(x + 2)(x - 2)(x - 4)} - \frac{(x - 4)}{(x + 2)(x - 2)(x - 4)}$$

Step 3. Subtract and simplify.

$$\frac{(x - 3)(x - 2) - (x - 4)}{(x + 2)(x - 2)(x - 4)} = \frac{(x^2 - 5x + 6) - (x - 4)}{(x + 2)(x - 2)(x - 4)} = \frac{x^2 - 6x + 10}{(x + 2)(x - 2)(x - 4)}$$

Lesson 11. Add or Subtract Rational Expressions

Practice 11.1. Add/subtract and simplify.

$$(1) \frac{x^2 + 2x - 2}{x^2 + 2x - 15} + \frac{5x + 12}{x^2 + 2x - 15}$$

$$(2) \frac{3x - 10}{x^2 - 25} - \frac{2x - 15}{x^2 - 25}$$

$$\frac{5-x}{1} \quad (2) \quad 1111$$

$$\frac{3-x}{2+x} \quad (1) \quad 1111$$

Practice 11.2. Find the LCD of rational expressions.

$$(1) \frac{2x}{2x^2 - 5x - 3} \quad \text{and} \quad \frac{x-1}{x^2 - x - 6}$$

$$(2) \frac{9}{7x^2 - 28x} \quad \text{and} \quad \frac{2}{x^2 - 8x + 16}$$

$$2(4-x)x \quad (2) \quad 1121$$

$$(2+x)(3-x)(1+x) \quad (1) \quad 1121$$

Practice 11.3. Add and simplify.

$$(1) \frac{x}{x+1} + \frac{x-1}{x+2}$$

$$(2) \frac{x+2}{2x^2 - x - 3} + \frac{1}{x^2 + 3x + 2}$$

$$(3) \frac{4}{x-3} + \frac{3x-2}{x^2 - x - 6}$$

$$\frac{(2+x)(3-x)}{9+7x} \quad (3) \quad 1131$$

$$\frac{(3-x)(2)(2+x)(1+x)}{1+x^2+6x+1} \quad (2) \quad 1131$$

$$\frac{(2+x)(1+x)}{2x^2+2x-1} \quad (1) \quad 1131$$

Practice 11.4. Subtract and simplify.

$$(1) \frac{3x+5}{x^2 - 7x + 12} - \frac{3}{x-3}$$

$$(2) \frac{y}{y^2 - 5y - 6} - \frac{7}{y^2 - 4y - 5}$$

$$(3) \frac{2x-3}{x^2 + 3x - 10} - \frac{x+2}{x^2 + 2x - 8}$$

$$\frac{(5+x)(4+x)(2-x)}{2x^2 - 2x - 22} \quad (3) \quad 1141$$

$$\frac{(y-1)(y-6)(y-9)(y+5)(y+1)}{y^2 - 12y + 42} \quad (2) \quad 1141$$

$$\frac{(x-4)(x-1)(x-3)}{17} \quad (1) \quad 1141$$



Lesson 12. Complex Rational Expressions

Simplify Complex Rational Expressions

A **complex rational expression** is a rational expression whose denominator or numerator contains a rational expression.

A complex rational expression is equivalent to the quotient of its numerator by its denominator. That suggests the following strategy to simplify a complex rational expression.

Step 1. Simplify the numerator and the denominator.

Step 2. Rewrite the expression as the numerator multiplying the reciprocal of the denominator.

Step 3. Multiply and simplify.

Example 12.1. Simplify

$$\frac{\frac{2x-1}{x^2-1} + \frac{x-1}{x+1}}{\frac{x+1}{x-1} - \frac{1}{x^2-1}}$$

Step 1. Simplify the numerator and the denominator.

$$\begin{aligned} \frac{\frac{2x-1}{x^2-1} + \frac{x-1}{x+1}}{\frac{x+1}{x-1} - \frac{1}{x^2-1}} &= \frac{\frac{2x-1}{(x-1)(x+1)} + \frac{(x-1)(x-1)}{(x-1)(x+1)}}{\frac{(x+1)(x+1)}{(x-1)(x+1)} - \frac{1}{(x-1)(x+1)}} = \frac{\frac{(2x-1) + (x-1)(x-1)}{(x-1)(x+1)}}{\frac{(x+1)(x+1) - 1}{(x-1)(x+1)}} \\ &= \frac{(2x-1) + (x^2 - 2x + 1)}{(x-1)(x+1)} = \frac{x^2}{(x-1)(x+1)} \\ &= \frac{(x^2 + 2x + 1) - 1}{(x-1)(x+1)} = \frac{x^2 + 2x}{(x-1)(x+1)} \end{aligned}$$

Step 2. Rewrite as a product.

$$\frac{\frac{x^2}{(x-1)(x+1)}}{\frac{x^2 + 2x}{(x-1)(x+1)}} = \frac{x^2}{(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{x^2 + 2x}$$

Step 3. Multiply and simplify.

$$\frac{x^2}{(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{x^2 + 2x} = \frac{x \cdot x}{(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{x(x+2)} = \frac{\cancel{x} \cancel{x} \cancel{(x-1)} \cancel{(x+1)}}{\cancel{x} (x+2) \cancel{(x-1)} \cancel{(x+1)}} = \frac{x}{x+2}$$

Lesson 12. Complex Rational Expressions

Practice 12.1. Simplify.

$$(1) \frac{1 + \frac{2}{x}}{1 - \frac{2}{x}}$$

$$(2) \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - \frac{1}{x}}$$

1 + x (2) 1.21

$$\frac{2-x}{2+x} \quad (1) \quad 1.21$$

Practice 12.2. Simplify.

$$(1) \frac{\frac{x^2 - y^2}{y^2}}{\frac{1}{x} - \frac{1}{y}}$$

$$(2) \frac{\frac{2}{(x+1)^2} - \frac{1}{x+1}}{1 - \frac{4}{(x+1)^2}}$$

$$\frac{3-x}{1} \quad (2) \quad 1.2.2$$

$$\frac{x}{(x+y)x} \quad (1) \quad 1.2.2$$

Practice 12.3. Simplify.

$$(1) \frac{\frac{5x}{x^2 - x - 6}}{\frac{2}{x+2} + \frac{3}{x-3}}$$

$$(2) \frac{\frac{x+1}{x-1} + \frac{x-1}{x+1}}{\frac{x+1}{x-1} - \frac{x-1}{x+1}}$$

$$\frac{2x}{x^2+1} \quad (2) \quad 1.2.3$$

1 (1) 1.2.3



Lesson 13. Rational Equations

Solve Rational Equations by Clearing Denominators

A **rational equation** is an equation that contains a rational expression.

One idea to solve a rational equation is to reduce the equation to a polynomial equation by clearing denominators, that is multiplying the LCD to both sides of the equation.

Step 1. Find the LCD.

Step 2. Clear denominators by multiplying the LCD to both sides.

Step 3. Solve the resulting polynomial equation.

Step 4. Check if there is an **extraneous solution** which is a solution that would cause any of the expressions in the original equation to be undefined.

Example 13.1. Solve

$$\frac{5}{x^2 - 9} = \frac{3}{x - 3} - \frac{2}{x + 3}.$$

Step 1. Find the LCD.

Since $x^2 - 9 = (x + 3)(x - 3)$, the LCD is $(x + 3)(x - 3)$.

Step 2. Clear denominators.

Multiply each rational expression in both sides by $(x + 3)(x - 3)$ and simplify:

$$\begin{aligned}(x + 3)(x - 3) \cdot \frac{5}{x^2 - 9} &= (x + 3)(x - 3) \cdot \frac{3}{x - 3} - (x + 3)(x - 3) \cdot \frac{2}{x + 3} \\ 5 &= 3(x + 3) - 2(x - 3)\end{aligned}$$

Step 3. Solve the resulting equation.

$$\begin{aligned}5 &= 3(x + 3) - 2(x - 3) \\ 5 &= x + 15 \\ -10 &= x\end{aligned}$$

Step 4. Check for any extraneous solution by plugging the solution into the LCD to see if it is zero. If it is zero, then the solution is extraneous.

$$(-10 + 3)(-10 - 3) \neq 0$$

So $x = -10$ is a valid solution of the original equation.

Example 13.2. Solve for x from the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}.$$

Step 1. The LCD is xyz .

Step 2. Clear denominators.

$$\begin{aligned}xyz \cdot \frac{1}{x} + xyz \cdot \frac{1}{y} &= xyz \cdot \frac{1}{z} \\ yz + xz &= xy\end{aligned}$$

Step 3. Solve the resulting equation.

$$\begin{aligned}yz + xz &= xy \\ yz &= xy - xz \\ yz &= x(y - z) \\ \frac{yz}{y - z} &= x\end{aligned}$$

Step 4. The solution is $x = \frac{yz}{y - z}$.

Lesson 13. Rational Equations

Practice 13.1. Solve.

$$(1) \frac{1}{x+1} + \frac{1}{x-1} = \frac{4}{x^2-1}$$

$$(2) \frac{30}{x^2-25} = \frac{3}{x+5} + \frac{2}{x-5}$$

$$L = x \quad (2) \quad 13.1$$

$$2 = x \quad (1) \quad 13.1$$

Practice 13.2. Solve.

$$(1) \frac{2x-1}{x^2+2x-8} = \frac{1}{x-2} - \frac{2}{x+4}$$

$$(2) \frac{3x}{x-5} = \frac{2x}{x+1} - \frac{42}{x^2-4x-5}$$

$$L = x \text{ or } -6 = x \quad (2) \quad 13.2$$

$$3 = x \quad (1) \quad 13.2$$

Practice 13.3. Solve a variable from a formula.

$$(1) \text{ Solve for } f \text{ from } \frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

$$(2) \text{ Solve for } x \text{ from } A = \frac{f+cx}{x}.$$

$$\frac{c-V}{f} = x \quad (2) \quad 13.3$$

$$\frac{b+d}{bd} = f \quad (1) \quad 13.3$$



Radical Expressions

If $b^2 = a$, then we say that b is a **square root** of a . We denote the positive square root of a as \sqrt{a} , called the **principal square root**.

For any real number a , the expression $\sqrt{a^2}$ can be simplified as

$$\sqrt{a^2} = |a|.$$

If $b^3 = a$, then we say that b is a **cube root** of a . The cube root of a real number a is denoted by $\sqrt[3]{a}$.

For any real number a , the expression $\sqrt[3]{a^3}$ can be simplified as

$$\sqrt[3]{a^3} = a.$$

In general, if $b^n = a$, then we say that b is an **n -th root** of a . If n is **even**, the **positive n -th root** of a , called the **principal n -th root**, is denoted by $\sqrt[n]{a}$. If n is odd, the n -th root $\sqrt[n]{a}$ of a has the same sign with a .

In $\sqrt[n]{a}$, the symbol $\sqrt{\quad}$ is called the **radical sign**, a is called the **radicand**, and n is called the **index**. If n is even, then the n -th root of a negative number is not a real number.

For any real number a , the expression $\sqrt[n]{a^n}$ can be simplified as

1. $\sqrt[n]{a^n} = |a|$ if n is even.
2. $\sqrt[n]{a^n} = a$ if n is odd.

Example 14.1. Simplify the radical expression using the definition.

$$(1) \sqrt{4(y-1)^2} \qquad (2) \sqrt[3]{-8x^3y^6}$$

Solution:

$$(1) \sqrt{4(y-1)^2} = \sqrt{[2(y-1)]^2} = 2|y-1|.$$

$$(2) \sqrt[3]{-8x^3y^6} = \sqrt[3]{(-2xy^2)^3} = -2xy^2.$$

Definition and Properties of Rational Exponents

If $\sqrt[n]{a}$ is a real number, then we define $a^{\frac{m}{n}}$ as

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

Rational exponents have the same properties as integral exponents:

$$\begin{array}{lll}
 1. a^m \cdot a^n = a^{m+n} & 2. \frac{a^m}{a^n} = a^{m-n} & 3. a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} \\
 4. (a^m)^n = a^{mn} & 5. (ab)^m = a^m \cdot b^m & 6. \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}
 \end{array}$$

Example 14.2. Simplify the radical expression or the expression with rational exponents. Write in radical notation.

$$(1) \sqrt{x} \sqrt[3]{x^2} \qquad (2) \sqrt[3]{\sqrt{x^3}} \qquad (3) \left(\frac{x^{\frac{1}{2}}}{x^{-\frac{5}{6}}}\right)^{\frac{1}{4}}$$

Solution:

$$(1) \sqrt{x} \sqrt[3]{x^2} = x^{\frac{1}{2}} x^{\frac{2}{3}} = x^{\frac{1}{2} + \frac{2}{3}} = x^{\frac{7}{6}} = x \sqrt[6]{x}.$$

$$(2) \sqrt[3]{\sqrt{x^3}} = (\sqrt{x^3})^{\frac{1}{3}} = [(x^3)^{\frac{1}{2}}]^{\frac{1}{3}} = x^{3 \cdot \frac{1}{2} \cdot \frac{1}{3}} = x^{\frac{1}{2}} = \sqrt{x}.$$

$$(3) \left(\frac{x^{\frac{1}{2}}}{x^{-\frac{5}{6}}}\right)^{\frac{1}{4}} = (x^{\frac{1}{2} + \frac{5}{6}})^{\frac{1}{4}} = (x^{\frac{1}{2} + \frac{5}{6}})^{\frac{1}{4}} = (x^{\frac{4}{3}})^{\frac{1}{4}} = x^{\frac{1}{3}} = \sqrt[3]{x}.$$

Lesson 14. Radical Expressions: Concepts and Properties

Practice 14.1. Evaluate the square root. If the square root is not a real number, state so.

(1) $-\sqrt{16}$

(2) $\sqrt{\frac{4}{25}}$

(3) $\sqrt{6} + \sqrt{49}$

(4) $-\sqrt{-1}$

14.1 (4) not a real number

14.1 (3) 10

14.1 (2) $\frac{5}{2}$

14.1 (1) -4

Practice 14.2. Simplify the radical expression.

(1) $\sqrt{(-7)^2}$

(2) $\sqrt{(x+2)^2}$

(3) $\sqrt{25x^2y^6}$

(4) $\sqrt[3]{-27x^3}$

(5) $\sqrt[4]{16x^8}$

(6) $\sqrt[5]{(2x-1)^5}$

14.2 (1) 7 (2) $|x+2|$ (3) $|5xy^3|$ (4) $-3x$ (5) $2x^2$ (6) $2x-1$

Practice 14.3. Write the radical expression with rational exponents.

(1) $\sqrt[5]{(2x)^5}$

(2) $(\sqrt[5]{3xy})^7$

14.3 (2) $(3xy)^{\frac{7}{5}}$

14.3 (1) $(2x)^{\frac{5}{5}}$



Lesson 14. Radical Expressions: Concepts and Properties

Practice 14.4. Write in radical notation and simplify.

(1) $4^{\frac{3}{2}}$

(2) $-81^{\frac{3}{4}}$

$14.4 (2) -27$

$14.4 (1) 8$

Practice 14.5. Simplify the expression. Write with positive rational exponents. Assume all variables represent nonnegative numbers.

(1) $\frac{12x^{\frac{1}{2}}}{4x^{\frac{2}{3}}}$

(2) $(x^{-\frac{3}{5}}y^{\frac{1}{2}})^{\frac{1}{3}}$

$14.5 (2) \frac{y^{\frac{1}{6}}}{x^{\frac{1}{5}}}$

$14.5 (1) \frac{x}{3}$

Practice 14.6. Simplify the expression. Write in radical notation. Assume all variables represent nonnegative numbers.

(1) $\sqrt[4]{16a^{12}y^2}$

(2) $\frac{\sqrt{x}}{\sqrt[3]{x}}$

(3) $\sqrt{\sqrt[3]{x}}$

(4) $\sqrt{x} \sqrt[3]{x}$

$14.6 (6) \sqrt[6]{x^5}$

$14.6 (5) \sqrt[6]{x}$

$14.6 (4) \sqrt[6]{x}$

$14.6 (3) \sqrt[6]{x^3}$

$14.6 (2) \frac{y^{\frac{1}{6}}}{x^{\frac{1}{12}}}$

$14.6 (1) \frac{x}{3}$



Lesson 15. Algebra of Radicals

Product and Quotient Rules for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

$$\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}.$$

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $b \neq 0$, then

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Example 15.1. Simplify the expression.

$$(1) \sqrt[4]{8xy^4} \sqrt[3]{2x^7y} \qquad (2) \frac{\sqrt[5]{96x^9y^3}}{\sqrt[5]{3x^{-1}y}}$$

Solution:

$$(1) \sqrt[4]{8xy^4} \sqrt[3]{2x^7y} = \sqrt[4]{(8xy^4) \cdot (2x^7y)} = \sqrt[4]{16x^8y^5} = \sqrt[4]{2^4(x^2)^4y^4 \cdot y} = 2x^2y \sqrt[4]{y}.$$

$$(2) \frac{\sqrt[5]{96x^9y^3}}{\sqrt[5]{3x^{-1}y}} = \sqrt[5]{\frac{96x^9y^3}{3x^{-1}y}} = \sqrt[5]{32x^{10}y^2} = \sqrt[5]{(2x^2)^5 \cdot y^2} = 2x^2 \sqrt[5]{y^2}.$$

Add/Subtract by Combining Like Radicals

Two radicals are called **like radicals** if they have the same index and the same radicand. We add or subtract like radicals by combining their coefficients.

Example 15.2. Simplify the expression.

$$\sqrt{8x^3} - \sqrt{(-2)^2x^4} + \sqrt{2x^5}.$$

Solution:

$$\sqrt{8x^3} - \sqrt{(-2)^2x^4} + \sqrt{2x^5} = 2x\sqrt{2x} - 2x^2 + x^2\sqrt{2x} = (x^2 + 2x)\sqrt{2x} - 2x^2.$$

Multiplying Radicals with Two or More Terms

Multiplying radical expressions with many terms is similar to that multiplying polynomials with many terms.

Example 15.3. Simplify the expression.

$$(\sqrt{2x} + 2\sqrt{x})(\sqrt{2x} - 3\sqrt{x}).$$

Solution:

$$\begin{aligned} (\sqrt{2x} + 2\sqrt{x})(\sqrt{2x} - 3\sqrt{x}) &= \sqrt{2x} \cdot \sqrt{2x} - 3\sqrt{x} \cdot \sqrt{2x} + 2\sqrt{x} \cdot \sqrt{2x} - 6\sqrt{x} \cdot \sqrt{x} \\ &= 2x - 3x\sqrt{2} + 2x\sqrt{2} - 6x = -4x - x\sqrt{2} = -(4 + \sqrt{2})x. \end{aligned}$$

Rationalizing Denominators

Rationalizing denominator means rewriting a radical expression into an equivalent expression in which the denominator no longer contains radicals.

Example 15.4. Rationalize the denominator.

$$(1) \frac{1}{2\sqrt{x^3}} \qquad (2) \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$$

Solution:

(1) In this case, to get rid of the radical in the bottom, we multiply the expression by $\frac{\sqrt{x}}{\sqrt{x}}$ so that the radicand in the bottom becomes a perfect power. $\frac{1}{2\sqrt{x^3}} = \frac{1}{2\sqrt{x^3}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{2\sqrt{x^2}\sqrt{x}} = \frac{\sqrt{x}}{2x^2}$.

(2) In this case, we use the formula $(a - b)(a + b) = a^2 + b^2$. Multiply the expression by $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$.

$$\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} = \frac{(\sqrt{x} + \sqrt{y})^2}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} = \frac{x + y + 2\sqrt{xy}}{x - y}.$$

Lesson 15. Algebra of Radicals

Practice 15.1. Multiply.

(1) $\sqrt[3]{4}\sqrt[3]{5}$

(2) $\sqrt{x+7}\sqrt{x-7}$ (Assume that $x \geq 7$.)

$\sqrt{49 - x^2}$ (2) 15.1

$\sqrt[3]{27}$ (1) 15.1

Practice 15.2. Simplify the radical expression. Assume all variables are positive.

(1) $\sqrt{50}$

(2) $\sqrt[3]{-8x^2y^3}$

(3) $\sqrt[5]{32x^{12}y^2z^8}$

(4) $\sqrt{20xy} \cdot \sqrt{4xy^2}$

(5) $\sqrt[3]{16} \cdot 5\sqrt[3]{2}$

(6) $\sqrt[5]{8x^4y^3z^3} \cdot \sqrt[5]{8xy^4z^8}$

$\sqrt[5]{2xy^2z^2} \sqrt[5]{2xy^2z^2}$ (9) 15.2

$\sqrt[3]{4} \sqrt[3]{10}$ (5) 15.2

$\sqrt[5]{2} \sqrt[5]{4xy^4}$ (4) 15.2

$\sqrt[5]{2x^2y^2z^2} \sqrt[5]{2x^2y^2z^2}$ (3) 15.2

$\sqrt[3]{2x^2y^2}$ (2) 15.2

$\sqrt[5]{2}$ (1) 15.2

Practice 15.3. Add or subtract. Assume all variables are positive. Answers must be simplified.

(1) $5\sqrt{6} + 3\sqrt{6}$

(2) $4\sqrt{20} - 2\sqrt{5}$

(3) $3\sqrt{32x^2} + 5x\sqrt{8}$

(4) $7\sqrt{4x^2} + 2\sqrt{25x} - \sqrt{16x}$

(5) $5\sqrt[3]{x^2y} + \sqrt[3]{27x^5y^4}$

(6) $3\sqrt{9y^3} - 3y\sqrt{16y} + \sqrt{25y^3}$

$\sqrt[5]{2x^2y^2}$ (3) 15.3

$\sqrt[5]{2x^2y^2} \sqrt[5]{2x^2y^2}$ (6) 15.3

$\sqrt[5]{2} \sqrt[5]{4xy^4}$ (4) 15.3



Lesson 15. Algebra of Radicals

Practice 15.4. Divide. Assume all variables are positive. Answers must be simplified.

(1) $\sqrt{\frac{9x^3}{y^8}}$

(2) $\sqrt[3]{\frac{32x^4}{x}}$

(3) $\frac{\sqrt{40x^5}}{\sqrt{2x}}$

(4) $\frac{\sqrt[3]{24a^6b^4}}{\sqrt[3]{3b}}$

15.4 (4) $2a^2b$

15.4 (3) $2x^2\sqrt[3]{5}$

15.4 (2) $2x\sqrt[3]{4}$

15.4 (1) $\frac{x^4}{\sqrt[3]{x}}$

Practice 15.5. Multiply and simplify. Assume all variables are positive.

(1) $\sqrt{2}(3\sqrt{3} - 2\sqrt{2})$

(2) $(\sqrt{5} + \sqrt{7})(3\sqrt{5} - 2\sqrt{7})$

(3) $(\sqrt{3} + \sqrt{2})^2$

(4) $(\sqrt{9} - \sqrt{5})(\sqrt{9} + \sqrt{5})$

(5) $(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)$

(6) $(2\sqrt[3]{x} + 6)(\sqrt[3]{x} + 1)$

15.5 (6) $2\sqrt[3]{x^2} + 8\sqrt[3]{x} + 9$

15.5 (3) $5 + 2\sqrt{6}$

15.5 (2) $1 + \sqrt{35}$

15.5 (5) x

15.5 (1) $3\sqrt{6} - 4$

Practice 15.6. Simplify the radical expression and rationalize the denominator. Assume all variables are positive.

(1) $\sqrt[3]{\frac{2}{25}}$

(2) $\sqrt{\frac{2x}{7y}}$

(3) $\frac{\sqrt[3]{x}}{\sqrt[3]{3y^2}}$

(4) $\frac{3x}{\sqrt[4]{x^3y}}$

(5) $\frac{6\sqrt{3}}{\sqrt{3}-1}$

(6) $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

(7) $\frac{3+\sqrt{2}}{2+\sqrt{3}}$

(8) $\frac{2}{\sqrt{x}-\sqrt{y}}$

15.6 (8) $\frac{x-x}{2\sqrt{x+2}\sqrt{y}}$

15.6 (7) $6 + 2\sqrt{2} - 3\sqrt{3} - 6\sqrt{3}$

15.6 (6) $4 + \sqrt{15}$

15.6 (5) $9 + 3\sqrt{3}$

15.6 (4) $\frac{x}{3\sqrt[4]{x^3y}}$

15.6 (3) $\frac{3y}{\sqrt[3]{9xy}}$

15.6 (2) $\frac{xy}{\sqrt{14xy}}$

15.6 (1) $\frac{5}{\sqrt[3]{10}}$

Lesson 16. Solve Radical Equations

Solve Radical Equations by Taking the n -th Power

The idea to solve a radical equation $\sqrt[n]{X} = a$ is to first take n -th power of both sides to get rid of the radical sign, that is $X = a^n$ and then solve the resulting equation.

Note: In general, before applying the above idea, a radical expression has to be isolated first.

Example 16.1. Solve the equation $x - \sqrt{x+1} = 1$.

Solution:

Step 1. Arrange terms so that one radical is isolated on one side of the equation.

$$x - 1 = \sqrt{x+1}$$

Step 2. Square both sides to eliminate the square root.

$$(x - 1)^2 = x + 1$$

Step 3. Solve the resulting equation.

$$x^2 - 2x + 1 = x + 1$$

$$x^2 - 3x = 0$$

$$x(x - 3) = 0$$

$$x = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 0 \quad \text{or} \quad x = 3$$

Step 4. Check all proposed solutions. Plug $x = 0$ into the original equation, we see that the left hand side is $0 - \sqrt{0+1} = 0 - \sqrt{1} = 0 - 1 = -1$ which is not equal to the right hand side. So $x = 0$ cannot be a solution.

Plug $x = 3$ into the original equation, we see that the left hand side is $3 - \sqrt{3+1} = 3 - \sqrt{4} = 3 - 2 = 1$. So $x = 3$ is a solution.

Example 16.2. Solve the equation $\sqrt{x-1} - \sqrt{x-6} = 1$.

Solution:

Step 1. Isolated one radical.

$$\sqrt{x-1} = \sqrt{x-6} + 1$$

Step 2. Square both sides to remove radical sign and then isolate the remaining radical.

$$x - 1 = (x - 6) + 2\sqrt{x-6} + 1$$

$$x - 1 = x - 5 + 2\sqrt{x-6}$$

$$4 = 2\sqrt{x-6}$$

$$2 = \sqrt{x-6}$$

Step 3. Square both sides to remove the radical sign and then solve.

$$\sqrt{x-6} = 2$$

$$x - 6 = 4$$

$$x = 10.$$

Since $10 - 1 > 0$ and $10 - 6 > 0$, $x = 10$ is a valid solution. Indeed,

$$\sqrt{10-1} - \sqrt{10-6} = \sqrt{9} - \sqrt{4} = 3 - 2 = 1.$$

Example 16.3. Solve the equation $-2\sqrt[3]{x-4} = 6$.

Solution:

Step 1. Isolated the radical.

$$\sqrt[3]{x-4} = -3$$

$$x - 4 = (-3)^3$$

$$x - 4 = -27$$

$$x = 23$$

Step 2. Cube both sides to eliminate the cube root and then solve the resulting equation.

The solution is $x = 23$.

Lesson 16. Solve Radical Equations

Practice 16.1. Solve each radical equation.

(1) $\sqrt{3x+1} = 4$

(2) $\sqrt{2x-1} - 5 = 0$

(3) $\sqrt{5x+1} = x+1$

(4) $x = \sqrt{3x+7} - 3$

(5) $\sqrt{6x+7} - x = 2$

(6) $\sqrt{x+2} + \sqrt{x-1} = 3$

(7) $\sqrt{x+5} - \sqrt{x-3} = 2$

(8) $3\sqrt[3]{3x-1} = 6$

16.1 (1) $x = 5$

16.1 (2) $x = 13$

16.1 (3) $x = 0$

16.1 (4) $x = -2$

16.1 (5) $x = -1$

16.1 (6) $x = 2$

16.1 (7) $x = 4$

16.1 (8) $x = 3$



Lesson 17. Complex Numbers

Complex Numbers

The imaginary unit i is defined as $i = \sqrt{-1}$. Hence $i^2 = -1$.

If b is a positive number, then $\sqrt{-b} = i\sqrt{b}$.

Let a and b are two real numbers. We define a complex number by the expression $a + bi$. The number a is called the real part and the number b is called the imaginary part. If $b = 0$, then the complex number $a + bi = a$ is just the real number. If $b \neq 0$, then we call the complex number $a + bi$ an imaginary number. If $a = 0$ and $b \neq 0$, then the complex number $a + bi = bi$ is called a purely imaginary number.

Adding, subtracting, multiplying, dividing or simplifying complex numbers are similar to those for radical expressions. In particular, adding and subtracting become similar to combining like terms.

Example 17.1. Simplify and write your answer in the form $a + bi$, where a and b are real numbers and i is the imaginary unit.

(1) $\sqrt{-3}\sqrt{-4}$ (2) $(4i - 3)(-2 + i)$ (3) $\frac{-2+5i}{i}$ (4) $\frac{1}{1-2i}$ (5) i^{2018}

Solution:

(1)

$$\sqrt{-3}\sqrt{-4} = i\sqrt{3} \cdot i\sqrt{4} = i^2 \cdot \sqrt{3} \cdot 2 = -2\sqrt{3}.$$

(2)

$$\begin{aligned}(4i - 3)(-2 + i) &= 4i \cdot (-2) + 4i \cdot i + (-3) \cdot (-2) + (-3) \cdot i \\ &= -8i + (-4) + 6 + (-3i) = 2 - 11i.\end{aligned}$$

(3)

$$\begin{aligned}\frac{-2 + 5i}{i} &= \frac{(-2 + 5i)i}{i \cdot i} = \frac{-2i + 5i^2}{i^2} \\ &= \frac{-2i - 5}{-1} = 5 + 2i.\end{aligned}$$

(4)

$$\begin{aligned}\frac{1}{1 - 2i} &= \frac{(1 + 2i)}{(1 - 2i)(1 + 2i)} = \frac{1 + 2i}{1 - (2i)^2} \\ &= \frac{1 + 2i}{5} = \frac{1}{5} + \frac{2}{5}i.\end{aligned}$$

(5)

$$i^{2018} = i^{4 \cdot 504 + 2} = (i^4)^{504} \cdot i^2 = -1.$$

Lesson 17. Complex Numbers

Practice 17.1. Add, subtract, multiply complex numbers and write your answer in the form $a + bi$.

(1) $\sqrt{-2} \cdot \sqrt{-3}$

(2) $\sqrt{2} \cdot \sqrt{-8}$

(3) $(5 - 2i) + (3 + 3i)$

(4) $(2 + 6i) - (12 - 4i)$

(5) $(3 + i)(4 + 5i)$

(6) $(7 - 2i)(-3 + 6i)$

(7) $(3 - x\sqrt{-1})(3 + x\sqrt{-1})$

(8) $(2 + 3i)^2$

(8) $12i + 5 - 10i + 10i$

(2) $6 + 2x + 8 + i$

(9) $12i + 6 - 48i + 48i$

(5) $12i + 7 + 19i - 2\sqrt{2}$

Practice 17.2. Divide the complex number and write your answer in the form $a + bi$.

(1) $\frac{2i}{1+i}$

(2) $\frac{5-2i}{3+2i}$

(3) $\frac{2+3i}{3-i}$

(4) $\frac{4+7i}{-3i}$

(4) $\frac{3}{7} + \frac{3}{7}i - \frac{3}{4}$

(3) $\frac{10}{11} + \frac{10}{3}i$

(2) $\frac{13}{16} - \frac{13}{11}i$

(1) $1 + 1i$

Practice 17.3. Simplify the expression.

(1) $(-i)^8$

(2) i^{15}

(3) i^{2017}

(4) $\frac{1}{i^{2018}}$

(4) -1

(3) i

(2) $-i$

(1) 1



Lesson 18. Complete the Square

The Square Root Property

Suppose u is an algebraic expression and d is a real number. If $u^2 = d$, then $u = \sqrt{d}$ or $u = -\sqrt{d}$.

Complete the Square

For a binomial $x^2 + bx$, one can obtain a perfect square trinomial by adding $\left(\frac{b}{2}\right)^2$:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

This procedure is called completing the square.

Solve by Completing the Square

Example 18.1. Solve the equation $x^2 + 2x - 1 = 0$.

Solution:

Step 1. Isolate the constant.

$$x^2 + 2x = 1$$

Step 2. With $b = 2$, add $\left(\frac{2}{2}\right)^2$ to both sides to complete a square for the binomial $x^2 + bx$.

$$\begin{aligned}x^2 + 2x + \left(\frac{2}{2}\right)^2 &= 1 + \left(\frac{2}{2}\right)^2 \\ \left(x + \frac{2}{2}\right)^2 &= 1 + 1 \\ (x + 1)^2 &= 2\end{aligned}$$

Step 3. Solve the resulting equation using the square root property.

$$\begin{aligned}x + 1 &= \sqrt{2} & \text{or} & & x + 1 &= -\sqrt{2} \\ x &= -1 + \sqrt{2} & \text{or} & & x &= -2 - \sqrt{2}\end{aligned}$$

Note that the solution can also be written as $x = -1 \pm \sqrt{2}$.

Example 18.2. Solve the equation $-2x^2 + 8x - 9 = 0$.

Solution:

Step 1. Isolate the constant.

$$-2x^2 + 8x = 9$$

Step 2. Divide by -2 to rewrite the equation in $x^2 + Bx = C$ form

$$x^2 - 4x = -\frac{9}{2}$$

Step 3. With $b = -4$, add $\left(\frac{-4}{2}\right)^2 = 4$ to both sides to complete the square for the binomial $x^2 - 4x$.

$$\begin{aligned}x^2 - 4x + 4 &= -\frac{9}{2} + 4 \\ (x - 2)^2 &= -\frac{1}{2}\end{aligned}$$

Step 4. Solve the resulting equation and simplify.

$$\begin{aligned}x - 2 &= \frac{\mathbf{i}}{\sqrt{2}} & \text{or} & & x - 2 &= -\frac{\mathbf{i}}{\sqrt{2}} \\ x &= 2 + \frac{\sqrt{2}}{2}\mathbf{i} & \text{or} & & x &= 2 - \frac{\sqrt{2}}{2}\mathbf{i}\end{aligned}$$

The solutions are $x = 2 \pm \frac{\sqrt{2}}{2}\mathbf{i}$.

Lesson 18. Complete the Square

Practice 18.1. Solve the quadratic equation by the square root property.

(1) $4x^2 = 20$ (2) $2x^2 - 6 = 0$ (3) $(x - 3)^2 = 10$ (4) $4(x + 1)^2 + 25 = 0$

$\frac{z}{15} \mp 1 = x$ (4) 18.1 $01 \wedge \mp \varepsilon = x$ (3) 18.1 $\varepsilon \wedge \mp = x$ (2) 18.1 $\underline{\varepsilon} \wedge \mp = x$ (1) 18.1

Practice 18.2. Solve the quadratic equation *by completing the square*.

(1) $x^2 - 6x + 25 = 0$ (2) $x^2 + 4x - 3 = 0$ (3) $x^2 - 3x - 5 = 0$

(4) $x^2 + x - 1 = 0$

(5) $x^2 + 8x + 12 = 0$

(6) $3x^2 + 6x - 1 = 0$

18.2 (4) $x = \frac{z}{15} \mp 1$ (9) 18.2
18.2 (3) $x = \frac{z}{15} \mp 1$

18.2 (5) $x = -6 \text{ or } 9 = x$ (5) 18.2
18.2 (2) $x = -2 \mp \sqrt{7}$

18.2 (1) $x = 3 \mp 4i$
18.2 (4) $x = \frac{z}{15} \mp 1$



Lesson 19. Quadratic Formula

The Quadratic Formula

The solutions of a quadratic equation in the **standard form** $ax^2 + bx + c = 0$ with $a \neq 0$ are given by **the quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

- If $b^2 - 4ac > 0$, the equation has two real solutions.
- If $b^2 - 4ac = 0$, the equation has one real solution.
- If $b^2 - 4ac < 0$, the equation has two imaginary solutions (no real solutions).

Example 19.1. Determine the type and the number of solutions of the equation $(x-1)(x+2) = -3$.

Solution:

Step 1. Rewrite the equation in the form $ax^2 + bx + c = 0$.

$$\begin{aligned}(x-1)(x+2) &= -3 \\ x^2 + x + 1 &= 0\end{aligned}$$

Step 2. Find the values of a , b and c .

$$a = 1, b = 1 \text{ and } c = 1.$$

Step 3. Find the discriminant $b^2 - 4ac$.

$$b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 1 = -3.$$

The equation has two imaginary solutions.

Example 19.2. Solve the equation $2x^2 - 4x + 7 = 0$.

Solution:

Step 1. Find the values of a , b and c .

$$a = 2, b = -4 \text{ and } c = 7.$$

Step 2. Find the discriminant $b^2 - 4ac$.

$$b^2 - 4ac = (-4)^2 - 4 \cdot 2 \cdot 7 = 16 - 56 = -40.$$

Step 3. Apply the quadratic formula and simplify.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{-40}}{2 \cdot 2} = \frac{4 \pm 2\sqrt{10}i}{4} = 1 \pm \frac{\sqrt{10}}{2}i.$$

Example 19.3. Find the base and the height of a **triangle** whose base is three inches more than two times its height and has an area of 5 square inches. Round to the nearest tenth of an inch.

Solution:

Step 1. We may suppose the height is x inches. The base can be expressed as $2x + 3$ inches.

Step 2. By the area formula for a triangle, we have an equation.

$$\frac{1}{2}x(2x + 3) = 5.$$

Step 3. Rewrite the equation in $ax^2 + bx + c = 0$ form.

$$\begin{aligned}x(2x + 3) &= 10 \\ 2x^2 + 3x - 10 &= 0.\end{aligned}$$

Step 4. By the quadratic formula, we have $x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot (-10)}}{2 \cdot 2} = \frac{-3 \pm \sqrt{89}}{4}$. Since x can not be negative, $x = \frac{-3 + \sqrt{89}}{4} \approx 1.6$ and $2x + 3 \approx 6.2$.

The height and base of the triangle are approximately 1.6 inches and 6.2 inches respectively.

Lesson 19. Quadratic Formula

Practice 19.1. Determine the number and the type of solutions of the given equation.

(1) $x^2 + 8x + 3 = 0$

(2) $3x^2 - 2x + 4 = 0$

(3) $2x^2 - 4x + 2 = 0$

19.1 (1) Two real solutions 19.1 (2) Two imaginary solutions 19.1 (3) One real solution

Practice 19.2. Solve using the quadratic formula.

(1) $x^2 + 3x - 7 = 0$

(2) $2x^2 = -4x + 5$

(3) $2x^2 = x - 3$

19.2 (1) $x = \frac{-3 \pm \sqrt{37}}{2}$ 19.2 (2) $x = \frac{-2 \pm \sqrt{14}}{2}$ 19.2 (3) $x = \frac{1 \pm \sqrt{23}}{4}$

Practice 19.3. A **triangle** whose area is 7.5 square meters has a base that is one meter less than triple the height. Find the length of its base and height. Round to the nearest hundredth of a meter.

19.3 The height is $\frac{6}{1 + \sqrt{181}} \approx 2.41$ meters and the base is $\frac{2}{-1 + \sqrt{181}} \approx 6.23$ meters.

Practice 19.4. A **rectangular** garden whose length is 2 feet longer than its width has an area 66 square feet. Find the dimensions of the garden, rounded to the nearest hundredth of a foot.

19.4 The width is $\sqrt{67} - 1 \approx 7.19$ feet and the length is $\sqrt{67} + 1 \approx 9.19$ feet.



Lesson 20. Graphs of Quadratic Functions

The Graph of a Quadratic Function

The graph of a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, is called a parabola.

A quadratic function $f(x) = ax^2 + bx + c$ can be written in the form $f(x) = a(x - h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Step 1. The line $x = h = -\frac{b}{2a}$ is called the axis of symmetry of the parabola.

Step 2. The point $(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ is called the vertex of the parabola.

Minimum and Maximum of a Quadratic Function

Consider the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$.

- If $a > 0$, then the parabola opens upward and f has a minimum $f\left(-\frac{b}{2a}\right)$ at the vertex.
- If $a < 0$, then the parabola opens downward and f has a maximum $f\left(-\frac{b}{2a}\right)$ at the vertex.

Intercepts of a Quadratic Function

Consider the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$.

- The y -intercept is $(0, f(0)) = (0, c)$.
- The x -intercepts, if exist, are the solutions of the equation $ax^2 + bx + c = 0$.

Example 20.1. Does the function $f(x) = 2x^2 - 4x - 6$ have a maximum or minimum? Find it.

Solution:

Step 1. Since $a > 2$, the function opens upward and has a minimum.

Step 2. Find the line of symmetry $x = -\frac{b}{2a}$: $x = \frac{-(-4)}{2 \cdot 2} = 1$.

Step 3. Find the minimum by plugging $x = 1$ into the function f . The minimum is $f\left(-\frac{b}{2a}\right) = f(1) = 2 - 4 - 6 = -8$.

Example 20.2. Consider the function $f(x) = -x^2 + 3x + 6$. Find values of x such that $f(x) = 2$.

Solution:

Step 1. Set up the equation for x .

$$-x^2 + 3x + 6 = 2$$

Step 2. Solve the equation $-x^2 + 3x + 6 = 2$.

We get $x = -1$ or $x = 4$.

The values of x such that $f(x) = 2$ are -1 and 4 .

Example 20.3. A quadratic function f whose the vertex is $(1, 2)$ has a y -intercept $(0, -3)$. Find the equation that defines the function.

Solution:

Step 1. Write down the general form of f using only the vertex.

Quadratic functions with the vertex at $(1, 2)$ are defined by $y = a(x - 1)^2 + 2$, where a is a nonzero real number.

Step 2. Determine the unknown a using the remaining information.

Since $(0, -3)$ is on the graph of the function, the number a must satisfy the equation $-3 = a(0 - 1)^2 + 2$.

Step 3. Solving for a from the equation, we get $a = -5$.

The quadratic function f is given by $f(x) = -5(x - 1)^2 + 2$.

Lesson 20. Graphs of Quadratic Functions

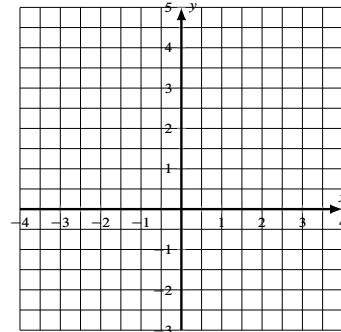
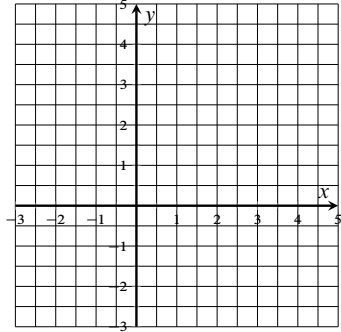
Practice 20.1. For the given quadratic function,

A. determine the coordinates of the x -intercepts, the coordinates of the y -intercept, the equation of the axis of symmetry and the coordinates of the vertex.

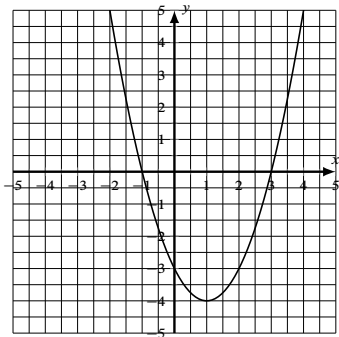
B. sketch the graph of the quadratic function using the information in part A.

(1) $f(x) = -(x - 2)^2 + 4$

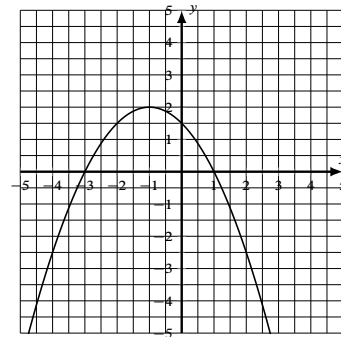
(2) $f(x) = x^2 + 2x - 3$



Practice 20.2. Consider the parabola in the graph below.



Practice 20.3. Consider the parabola in the graph below.



(1) For what values of x is y negative? Express your answer in interval notation.

(2) Find the domain of the function.

(3) Find the range of the function.

(4) Determine the coordinates of the x -intercepts.

(5) Determine the coordinates of the y -intercept.

(6) Determine the coordinates of the vertex.

(7) For what values of x is $f(x) = -3$.

(8) Find the equation of the function.

(1) For what values of x is y negative? Express your answer in interval notation.

(2) Find the domain of the function.

(3) Find the range of the function.

(4) Determine the coordinates of the x -intercepts.

(5) Determine the coordinates of the y -intercept.

(6) Determine the coordinates of the vertex.

(7) For what values of x is $f(x) = \frac{3}{2}$.

(8) Find the equation of the function.

Lesson 21. Rational and Radical Functions

The Domain of a Rational Function

A rational function f is defined by an equation $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and the degree of $q(x)$ is at least one. Since the denominator cannot be zero, the domain of f consists all real numbers except the numbers such that $q(x) = 0$

Example 21.1. Find the domain of the function $f(x) = \frac{1}{x-1}$.

Solution: Solve the equation $x - 1 = 0$, we get $x = 1$. Then the domain is $\{x \mid x \neq 1\}$. In interval notation, the domain is

$$(-\infty, 1) \cup (1, \infty).$$

The Domain of a Radical Function

A radical function f is defined by an equation $f(x) = \sqrt[n]{r(x)}$, where $r(x)$ is an algebraic expression. For example $f(x) = \sqrt{x+1}$. When n is odd number, $r(x)$ can be any real number. When n is even, $r(x)$ has to be nonnegative, that is $r(x) \geq 0$ so that $f(x)$ is a real number.

Example 21.2. Find the domain of the function $f(x) = \sqrt{x+1}$.

Solution: Since the index is 2 which is even, the function has real outputs only if the radicand $x + 1 \geq 0$. Solve the inequality, we get $x \geq -1$. In interval notation, the domain is

$$[-1, \infty).$$

Practice 21.1. Find the domain of each function.

(1) $f(x) = \frac{x^2}{x-2}$ (2) $f(x) = \frac{x}{x^2-1}$ (3) $f(x) = \sqrt{2x-3}$ (4) $f(x) = \sqrt{x^2+1}$

21.1.17 (1) $(-\infty, 2) \cup (2, \infty)$ (2) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ (3) $[\frac{3}{2}, \infty)$ (4) $(-\infty, \infty)$



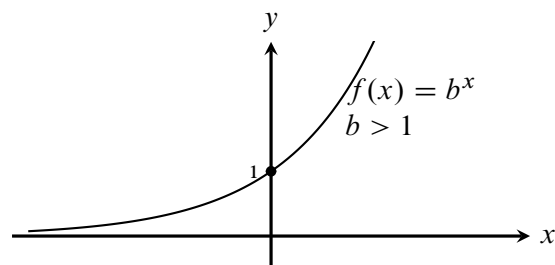
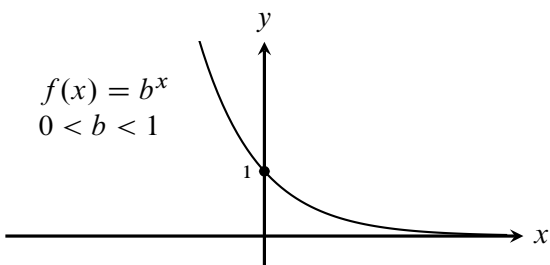
Lesson 22. Exponential Functions

Exponential Functions

Let b be a positive number other than 1 (i.e. $b > 0$ and $b \neq 1$). The exponential function f of x with the base b is defined as

$$f(x) = b^x \quad \text{or} \quad y = b^x.$$

Graphs of exponential functions:



The Natural Number e

The natural number e is the number to which the quantity $\left(1 + \frac{1}{n}\right)^n$ approaches as n takes on increasingly large values. Approximately, $e \approx 2.718281827$.

Compound Interests

After t years, the balance A in an account with a principal P and annual interest rate r is given by the following formulas:

1. For n compounding periods per year: $A = P \left(1 + \frac{r}{n}\right)^{nt}$.
2. For compounding continuously: $A = Pe^{rt}$.

Example 22.1. A sum of \$10,000 is invested at an annual rate of 8%, Find the balance, to the nearest hundredth dollar, in the account after 5 years if the interest is compounded

- (1) monthly, (2) quarterly, (3) semiannually, (4) continuous.

Solution:

Step 1. Find values of P , r , t and n . In this case, $P = 10,000$, $r = 8\% = 0.08$, $t = 5$ and n depends on compounding.

Step 2. Plug the values in the formula and calculate.

- (1) “Monthly” means $n = 12$. Then

$$A = 10000 \left(1 + \frac{0.08}{12}\right)^{5 \cdot 12} \approx 14898.46.$$

- (2) “Quarterly” means $n = 4$. Then

$$A = 10000 \left(1 + \frac{0.08}{4}\right)^{5 \cdot 4} \approx 14859.47.$$

- (3) “semiannually” means $n = 2$. Then

$$A = 10000 \left(1 + \frac{0.08}{2}\right)^{5 \cdot 2} \approx 14802.44.$$

- (4) For continuously compounded interest, we have

$$A = 10000e^{0.08 \cdot 5} \approx 14918.25.$$

Lesson 22. Exponential Functions

Practice 22.1. The value of a car is depreciating according to the formula: $V = 25000(3.2)^{-0.05x}$, where x is the age of the car in years. Find the value of the car, to the nearest dollar, when it is five years old.

22.1 (1) \$18692

Practice 22.2. A sum of \$20,000 is invested at an annual rate of 5.5%, Find the balance, to the nearest dollar, in the account after 5 years subject to

- (1) monthly compounding,
- (2) continuously compounding.

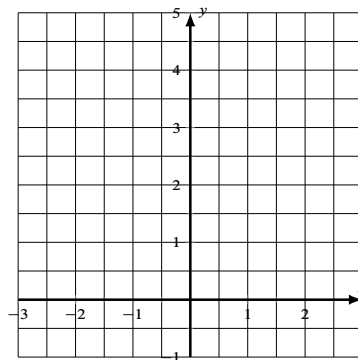
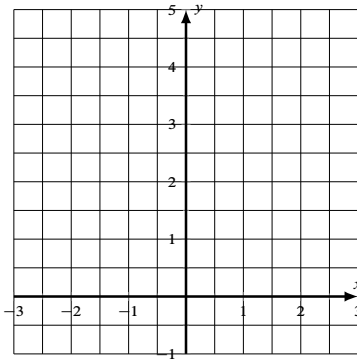
22.2 (2) \$26331

22.2 (1) \$26314

Practice 22.3. Sketch the graph of the function and find its range.

(1) $f(x) = 3^x$

(2) $f(x) = \left(\frac{1}{3}\right)^x$



Practice 22.4. Use the given function to compare the values of $f(-1.05)$, $f(0)$ and $f(2.4)$ and determine which value is the largest and which value is the smallest. Explain your answer.

(1) $f(x) = \left(\frac{5}{2}\right)^x$

(2) $f(x) = \left(\frac{2}{3}\right)^x$

22.4 (2) Largest: $f(-1.05)$; Smallest: $f(2.4)$.

22.4 (1) Largest: $f(2.4)$; Smallest: $f(-1.05)$.



Lesson 23. Logarithmic Functions

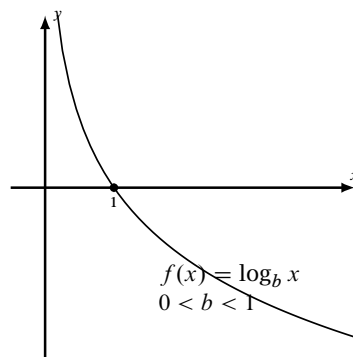
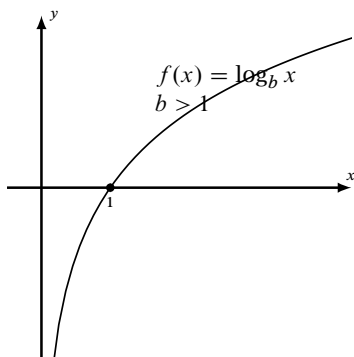
Logarithmic Function

For $x > 0$, $b > 0$ and $b \neq 1$, there is a unique number y satisfying the equation $b^y = x$. We denote the unique number y by $\log_b x$, read as logarithm to the base b of x . In other words, the defining relation between exponentiation and logarithm is

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

The function $f(x) = \log_b x$ is called the logarithmic function f of x with the base b .

Graphs of logarithmic functions:



Common Logarithms and Natural Logarithms

A logarithmic function $f(x)$ with base 10 is called the common logarithmic function and denoted by $f(x) = \log x$.

A logarithmic function $f(x)$ with base the natural number e is called the natural logarithmic function and denoted by $f(x) = \ln x$.

Basic Properties of Logarithms

When $b > 0$ and $b \neq 1$, and $x > 0$, we have

1. $b^{\log_b x} = x$.
2. $\log_b(b^x) = x$.
3. $\log_b b = 1$ and $\log_b 1 = 0$.

Example 23.1. Convert between exponential and logarithmic forms. (1) $\log x = \frac{1}{2}$ (2) $3^{2x-1} = 5$

Solution: Roughly speaking, when converting, we simply move the base from one side to the other side, then add or drop the log sign.

(1) Move the base 10 to the right side and drop the log from the left:

$$x = 10^{\frac{1}{2}}.$$

(2) Move the 3 to the right and add log the the right:

$$2x - 1 = \log_3 5.$$

Example 23.2. Evaluate the logarithms. (1) $\log_4 2$ (2) $10^{\log(\frac{1}{2})}$ (3) $\log_5(e^0)$

Solution: The key is to rewrite the log and the power so that they have the same base.

$$(1) \log_4 2 = \log_4 4^{\frac{1}{2}} = \frac{1}{2}.$$

$$(2) 10^{\log \frac{1}{2}} = 10^{\log_{10} \frac{1}{2}} = \frac{1}{2}$$

$$(3) \log_5(e^0) = \log_5 1 = 0$$

Example 23.3. Find the domain of the function $f(x) = \ln(2 - 3x)$.

Solution: The function has a real output if $2 - 3x > 0$. Solving the inequality, we get $x < \frac{2}{3}$. So the domain of the function is $(-\infty, \frac{2}{3})$.

Lesson 23. Logarithmic Functions

Practice 23.1. Write each equation into equivalent exponential form.

(1) $\log_3 7 = y$

(2) $3 = \log_b 64$

$49 = b^3$ (2) $3.1 = \log_2 64$

$7 = 3^x$ (1) $1.2 = \log_2 10$

Practice 23.2. Write each equation into equivalent logarithmic form.

(1) $7^x = 10$

(2) $b^5 = 2$

$5 = \log_2 2$ (2) $2.2 = \log_2 5$

$10 = \log_7 x$ (1) $2.2 = \log_2 10$

Practice 23.3. Evaluate.

(1) $\log_2 16$

(2) $\log_9 3$

(3) $\log 10$

(4) $\ln 1$

(5) $e^{\ln 2}$

(6) $\log 10^{\frac{1}{3}}$

(7) $\ln(\sqrt{e})$

(8) $\log_2(\frac{1}{2})$

2.3 (1) 4 2.3 (2) $\frac{2}{1}$ 2.3 (3) 1 2.3 (4) 0 2.3 (5) 2 2.3 (6) $\frac{3}{1}$ 2.3 (7) $\frac{2}{1}$ 2.3 (8) -1

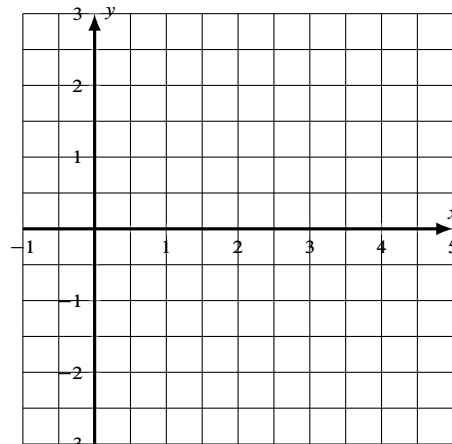
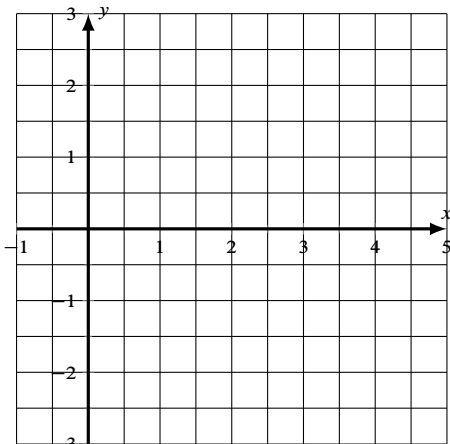
Practice 23.4. Find the domain of the function $f(x) = \log(x - 5)$. Write in interval notation.

$(5, +\infty)$ (5) $2.3.4$

Practice 23.5. Sketch the graph of each function and find its range.

(1) $f(x) = \log_2 x$

(2) $f(x) = \log_{\frac{1}{2}} x$



Lesson 24. Properties of Logarithms

Properties of Logarithms

For $M > 0$, $N > 0$, $b > 0$ and $b \neq 1$, we have

1. (The product rule) $\log_b(MN) = \log_b M + \log_b N$
2. (The quotient rule) $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$.
3. (The power rule) $\log_b(M^p) = p \log_b M$, where p is any real number.
4. (The change-of-base property) $\log_b M = \frac{\log_a M}{\log_a b}$, where $a > 0$ and $a \neq 1$. In particular,

$$\log_b M = \frac{\log M}{\log b} \quad \text{and} \quad \log_b M = \frac{\ln M}{\ln b}.$$

Example 24.1. Expand and simplify the logarithm $\log_2\left(\frac{8\sqrt{y}}{x^3}\right)$.

Solution:

$$\begin{aligned}\log_2\left(\frac{8\sqrt{y}}{x^3}\right) &= \log_2(8\sqrt{y}) - \log_2(x^3) \\ &= \log_2 8 + \log_2(y^{\frac{1}{2}}) - 3 \log_2 x \\ &= 3 + \frac{1}{2} \log_2 y - 3 \log_2 x.\end{aligned}$$

Example 24.2. Write the expression $2 \ln(x - 1) - \ln(x^2 + 1)$ as a single logarithm.

Solution:

$$2 \ln(x - 1) - \ln(x^2 + 1) = \ln((x - 1)^2) - \ln(x^2 + 1) = \ln\left(\frac{(x - 1)^2}{x^2 + 1}\right).$$

Example 24.3. Evaluate the logarithm $\log_3 4$ and round it to the nearest tenth.

Solution: On most scientific calculator, there are only the common logarithmic function **LOG** and the natural logarithmic function **LN**. To evaluate a logarithm based on a general number, we use the change-of-base property. In this case, the value of $\log_3 4$ is

$$\log_3 4 = \frac{\log 4}{\log 3} \approx 1.3.$$

Lesson 24. Properties of Logarithms

Practice 24.1. Expand the logarithm and simplify.

(1) $\log(100x)$

(2) $\ln\left(\frac{10}{e^2}\right)$

(3) $\log_b(\sqrt[3]{x})$

(4) $\log_7\left(\frac{x^2\sqrt{y}}{z}\right)$

$24.1 (1) 2 + \log x \quad 24.1 (2) \ln 10 - 2 \quad 24.1 (3) \frac{5}{1} \log_b x \quad 24.1 (4) 2 \log_7 x + \frac{7}{1} \log_7 y - \log_7 z$

Practice 24.2. Write as a single logarithm.

(1) $\frac{1}{3} \log x + \log y$

(2) $\frac{1}{2} \ln(x^2 + 1) - 2 \ln x$

(3) $3 \log_3 x - 2 \log_3(1 - x) + \frac{1}{3} \log_3(x^2 + 1)$.

$24.2 (3) \log_3 \left(\frac{x^3 \sqrt[3]{x^2+1}}{z(1-x)^2} \right)$

$24.2 (2) \ln \left(\frac{x^2}{\sqrt{x^2+1}} \right)$

$24.2 (1) \log \sqrt[3]{xy}$

Practice 24.3. Evaluate the logarithm and round it to the nearest hundredth.

(1) $\log_2 10 \approx$

(2) $\log_3 5 \approx$

$24.3 (2) 1.46$

$24.3 (1) 3.32$



Lesson 25. Exponential and Logarithmic Equations

Use Logarithms to Solve Exponential Equations

Step 1. Rewrite the equation in the form $b^u = c$, where u is an algebraic expression.

Step 2. Take logarithm of both sides with respect to the base b (or using the definition to convert).

Step 3. Simplify the resulting equation $u = \log_b c$ and solve for the variable.

Example 25.1. Solve the equation $10^{2x-1} - 5 = 0$.

Solution:

Step 1. Rewrite the equation in the form $b^u = c$: $10^{2x-1} = 5$.

Step 2. Take logarithm of both sides and simplify: $2x - 1 = \log 5$.

Step 3. Solve the resulting equation: $x = \frac{1}{2}(\log 5 + 1)$.

Use Exponential Form to Solve Logarithmic Equations

Step 1. Rewrite the equation in the form $\log_b u = c$, where u is an algebraic expression.

Step 2. Use the definition of logarithm to rewrite the equation in the form $u = b^c$.

Step 3. Solve the resulting equation.

Step 4. Check proposed solutions in the original equation.

Example 25.2. Solve the equation $\log_2 x + \log_2(x - 2) = 3$.

Solution:

Step 1. Rewrite the equation in the form $\log_b u = c$:

$$\log_2(x(x - 2)) = 3$$

Step 2. Rewrite the equation in the exponential form (moving the base):

$$x(x - 2) = 2^3$$

Step 3. Solve the resulting equation $x^2 - 2x - 8 = 0$. The solutions are $x = -2$ and $x = 4$

Step 4. Check proposed solutions. Both x and $x - 2$ has to be positive. So $x = -2$ is not a solution of the original equation. When $x = 4$, we have $\log_2 4 + \log_2 2 = 2 + 1 = 3$. So $x = 4$ is a solution.

Solve from Compound Interest Model

Example 25.3. A check of \$5000 was deposited in a savings account with an annual interest rate 6% which is compounded monthly. How many years will it take for the money to raise by 20%?

Solution: The question tells us the following information: $P = 5000$, $r = 0.06$, $n = 12$, and $A = 5000 \cdot (1 + 0.2) = 6000$. What we want to know is the number of years t . The compound interest model tells us that t satisfies the following equation:

$$6000 = 5000 \left(1 + \frac{0.06}{12}\right)^{12t}$$

This is an exponential equation and can be solve using logarithms.

$$5000 \left(1 + \frac{0.06}{12}\right)^{12t} = 6000$$

$$\left(1 + \frac{0.06}{12}\right)^{12t} = 1.2$$

$$12t \cdot \log(1 + 0.06 \div 12) = 1.2$$

$$12t = \log(1.2) \div \log(1 + 0.06 \div 12)$$

$$t = \log(1.2) \div \log(1 + 0.06 \div 12) \div 12 \approx 3.$$

So it takes about 3 years for the savings to raise by 20%.

Lesson 25. Exponential and Logarithmic Equations

Practice 25.1. Solve the exponential equation.

(1) $2^{x-1} = 4$

(2) $7e^{2x} - 5 = 58$

$x = 3$ (1) (2) 25.1

$x = 3$ (1) (2) 25.1

Practice 25.2. Solve the logarithmic equation.

(1) $\log_5 x + \log_5(4x - 1) = 1$

(2) $\ln \sqrt{x+1} = 1$

(3) $\log_2(x+2) - \log_2(x-5) = 3$

(4) $\log_3(x-5) = 2 - \log_3(x+3)$

$x = 6$ (4) 25.2

$x = 6$ (3) 25.2

$x = -1$ (2) 25.2

$x = \frac{4}{5}$ (1) 25.2

Practice 25.3. Using the formula $A = P(1 + \frac{r}{n})^{nt}$ to determine how many years, to the nearest hundredth, it will take to double an investment \$20,000 at the interest rate 5% compounded monthly.

25.3 14 years

